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EQUIVALENCE OF TOPOLOGICAL FORM FOR CURVILINEAR GEOMETRIC OBJECTS

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ABSTRACT

Given a curvilinear geometric object in R^3, made up of properly-joined parametric patches defined in terms of control points, it is of interest to know under what conditions the object will retain its original topological form when the control points are perturbed. For example, the patches might be triangular Bézier surface patches, and the geometric object may represent the boundary of a solid in a solid-modeling application. In this paper we give sufficient conditions guaranteeing that topological form is preserved by an ambient isotopy. The main conditions to be satisfied are that the original object should be continuously perturbed in a way that introduces no self-intersections of any patch, and such that the patches remain properly joined. The patches need only have C^0 continuity along the boundaries joining adjacent patches. The results apply directly to most surface modeling schemes, and they are of interest in several areas of application.

Keywords: geometric modeling, ambient isotopy, topological equivalence.

1. Introduction

Many problems in computational geometry involve the notion of equivalent topological form. For example, this concept is an intrinsic part of the formulation of *triangulation*, where a frequently used definition of sameness of topological form is that of homeomorphism.[1] This definition is also implicit in the problems of *graph isomorphism* [2] and *congruence of point sets*.[3] Similarly, the concept of equivalent topological form is used in many application fields of computational geometry. In particular, it is used in *computer graphics*,[4,5] *robotics*,[6] *image processing*,[7,8,9] *computer aided geometric design (CAGD)*,[10] and *solid modeling*.[11] It is the last two of the mentioned application fields that provide the primary motivation for this paper.

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Other fields that use the concept of equivalent topological form are tolerancing and metrology, engineering design, and finite-element analysis; references illustrating such use, in each of these fields, have been listed elsewhere.[12]

As previously observed,[12] the idea that we are willing to accept variation in a geometric object, but that we insist that it should retain its original topological form, has powerful intuitive appeal. The idea is often used only in an intuitive way, but rigorous definitions, which vary depending upon the application, have also been introduced: for example, homotopies [1,6] and homeomorphisms.[1] In the context of solid modeling and CAGD, a stringent (but, as we will show, practicable) requirement, for acceptance of two geometrical objects as topologically equivalent, is that they should be linked by an *ambient isotopy* in R^3. In this paper we show how to guarantee that an original and perturbed curvilinear simplicial complex are linked by an ambient isotopy. The theorem proved is a very general result which can easily be applied in many situations occurring in applications; for example, in the special case of complexes made up of triangular Bézier patches, sufficient conditions that the hypotheses of our theorem be satisfied have been presented elsewhere.[13]

The organization of the paper is as follows. In Section 2 we define ambient isotopy. In Section 3, we state the main result, and in Section 4, we give its proof. Section 5 concludes the paper.

2. Curvilinear Simplexes and Ambient Isotopy

Suppose that we are given a finite curvilinear simplicial complex [14] $K = \cup_{i=1}^{N} S_i$ in R^3 where each S_i is a two-dimensional patch defined in terms of its control points. For example, S_i might be a Bézier, B-spline or NURBS (Non-Uniform Rational B-spline) surface patch [15] defined in R^3, and K might represent the boundary of a three-dimensional solid (with curvilinear boundary) in a solid-modeling application. These patches will be *properly joined* if they are disjoint, or intersect along a common boundary curve or in a vertex point. If we suppose now that the control points of the S_i are perturbed, then we might ask under what conditions the complex K will retain its original topological form. The perturbations referred to here might, for example, correspond to small perturbations of the control points, caused by the use of finite-precision arithmetic or by interactive graphical editing. Thus, we introduce a parameter $t \in [0, 1]$, write $K(t) = \cup_{i=1}^{N} S_i(t)$, where $S_i(t)$ is a perturbed version of $S_i(0) = S_i$, $i = 1, \ldots, N$, and ask under what conditions $K(t)$ has the same topological form as $K(0) = K$.

We turn now to the definition of "same topological form". One interpretation of these words is that two sets should be linked by a homeomorphism.[1] However, we show that if perturbations of the control points are appropriately restricted, then the perturbed control points define new objects that are similar according to a much more stringent condition, namely, *ambient isotopy*, which informally means that there is an elastic motion of the ambient space R^3 that moves one object into a position congruent to the other.[16] An intuitive illustration of this condition can be given, as follows.[12] Imagine that the object is made of red putty, and that the rest of R^3 is filled with black putty; if the object *and* the surrounding space are elastically deformed, with no ripping or cutting of the boundary of the object, and without introducing new self-intersections of the boundary, then the modified object (the deformed red putty) has the same topological form as the original. Thus, a cube might be transformed into a solid spherical ball, but if the original and perturbed objects are to be of the same topological form, then *two disjoint rings cannot become interlinked* ,[4] *a torus cannot become a torus with a knot in it*, and *a trefoil knot cannot change orientation* .[14] The possibilities precluded in the conclusion of the previous sentence would be permitted if topological equivalence were defined merely by a homeomorphism between the original and perturbed object.

We now give the formal definition of an ambient isotopy.

Definition 1 *The complexes* $K(0)$ *and* $K(1)$ *are linked by an ambient isotopy if there is a continuous mapping*

$$F : [0,1] \times R^3 \mapsto R^3$$

such that for each $t \in [0,1]$, $F(t,\cdot)$ *is a homeomorphism from* R^3 *onto* R^3 *which carries* $K(0)$ *onto* $K(t)$.

Below, we will give sufficient conditions for the existence of an ambient isotopy linking $K(0)$ and $K(t)$.

3. Equivalence of Curvilinear Simplicial Complexes

Suppose there is a one-parameter family of homeomorphisms, depending continuously on the parameter, carrying $K(0)$ onto $K(t)$ and $S_i(0)$ onto $S_i(t)$ for all parameter values t, where the patches $S_i(0)$ themselves do not self-intersect. In this paper, we show that the hypotheses contained in Assumptions 1 and 2 below are sufficient to guarantee the existence of an *ambient isotopy* between $K(0)$ and $K(t)$.

Let α denote a control-point vector that varies over some open subset Ω in R^m. (More specific information about the form of this control-point vector will be given at the end of this section.) For each such α, let $S(\alpha)$ be a curvilinear triangular surface patch in R^3, represented by a mapping $X(\alpha; u, v) \in R^3$, where (u, v) is in some, arbitrary, closed triangle[a] $\mathcal{T}$ in parameter space, $\mathcal{T} \subset R^2$. We will assume that the mapping X is in $C^2(\Omega \times \mathcal{T})$, and, for each α, we will denote the mapping from $\mathcal{T}$ into R^3 by $X(\alpha)$. The 3×2 Jacobian of $X(\alpha)$ is denoted by $A(\alpha)$ and its value at a point $P = (u, v) \in \mathcal{T}$ by $A(\alpha; u, v)$ or $A(\alpha; P)$. Note that, for triangular parameter domains, $X(\alpha)$ may be considered as a function of the barycentric coordinates for (u, v) with respect to the corner points of $\mathcal{T}$, so that the surface patch is independent of the particular choice of the triangle $\mathcal{T}$ in parameter space. In many applications it is convenient to choose different domains $\mathcal{T}_i$ for different surface patches. This introduces no essential changes in the proof: we have used $\mathcal{T}$ throughout, for simplicity.

Now consider a finite collection, $S(\alpha_i)$, $1 \leq i \leq N$ of surface patches. We introduce the following notation.

[a]The domain $\mathcal{T}$ may also be a rectangle in R^2 or some other polygonal domain with strictly-convex corners. In that case the subsequent analysis will go through with only formal changes.

For each i the control-point vector α_i is a function $\alpha_i = \alpha_i(t)$ of a parameter $t \in [0, 1]$, with $\alpha_i(t) \in \Omega$ for all t. Subsequently we will denote the surface patches $S(\alpha_i(t))$ by $S_i(t)$, the mappings $X(\alpha_i(t); u, v)$ or $X(\alpha_i(t); P)$ by $X_i(t; u, v)$ or $X_i(t; P)$ and the Jacobians (with respect to (u, v)) $A(\alpha_i(t); u, v)$ or $A(\alpha_i(t); P)$ by $A_i(t; u, v)$ or $A_i(t; P)$.

We will now introduce a definition, and make the following assumptions.

Definition 2 *Assuming that the patches are properly joined, we say that two adjacent patches $S_i(t)$ and $S_j(t)$ are* nontangential *if for any two regular curves in $S_i(t)$ and $S_j(t)$ emanating from an arbitrary point $P \in S_i(t) \cap S_j(t)$, their tangent vectors at the point P are not parallel and identically directed, unless the patches intersect along a common curve and the tangent vectors are both parallel to the tangent vector of $S_i(t) \cap S_j(t)$ at the point P.*

Assumption 1 *For every patch $S_i(t)$, its control-point vector $\alpha_i = \alpha_i(t)$ is a continuously differentiable function of t.*

Assumption 2 *For all $t \in [0, 1]$ the patches $S_i(t)$ $1 \leq i \leq N$, are properly joined. Moreover, they are properly joined in the same way for every t. Also, the mappings $X_i(t) : \mathcal{T} \rightarrow R^3$ are one-to-one and their Jacobians $A_i(t; P)$ are non-singular[b] for all $P \in \mathcal{T}$ and all $t \in [0, 1]$. Finally, any two adjacent patches, $S_i(t)$ and $S_j(t)$ are nontangential for all t.*

Note that our assumptions impose no requirement on the level of continuity at patch intersections: we require only simple continuity, and that the intersection be non-tangential. The latter requirement, imposed in Assumption 2 and defined in Definition 2, may be rephrased informally as follows. If we imagine departing from a point in the intersection of two patches, and following along each of two regular curves, one in each patch, then the tangent vectors of the two regular curves must not be the same, except possibly in one case. The exception is the obvious one: if the two patches intersect along a boundary curve, then it is possible that the two regular curves, one in each patch, are in fact identical, with tangent vector equal to the tangent vector of the boundary curve.

We now state the main theorem, which is proved in Section 4.

Theorem 1 *Under assumptions 1 and 2, the sets $K(t)$ are linked by an ambient isotopy, F.*

For the particular case where each $S_i(t)$ is a triangular non-selfintersecting Bézier patch with non-singular Jacobian, and whose perturbation results from perturbation of its control points, we have given sufficient conditions [13] to satisfy Assumption 2.

Although the proof of Theorem 1 is deferred until Section 4, some comments are in order here. First of all, Assumption 1 can be replaced by the weaker condition that $\alpha_i(t)$ is only a continuous function of t: we have assumed continuous differentiability in order to simplify the proofs. Assumption 2 excludes, in particular, the case of a cusp formed by two adjacent patches.

Assumption 2 implies that the complexes $K(t)$ are homeomorphic for all $t \in [0, 1]$, but we establish the stronger conclusion of the existence of ambient iso-

[b]The 3×2 Jacobian is non-singular if it is of full rank.

topy. The literature contains many theorems for extending an isotopy to an ambient isotopy. Representative examples include an elementary proof for piecewise-linear simplicial complexes,[12] theorems giving necessary and sufficient conditions on piecewise-linear compact polyhedra,[17] and theorems involving restrictions to C^∞ compact submanifolds.[18] Theorem 1 of this paper relies upon weaker hypotheses than any of these cited results.

We also state a corollary to Theorem 1.

Corollary 1 *Given any open set $\mathcal{O}$ containing the set $\bigcup_{0\le t\le 1} K(t)$, then the ambient isotopy F may be chosen such that, for $0 \le t \le 1, F(t,\cdot)$ equals the identity mapping outside $\mathcal{O}$.*

Corollary 1 expresses the intuitively obvious notion that in order to transform $K(0)$ into $K(1)$, then only some arbitrarily small amount of additional "elbow room" must also be deformed about the region in R^3 through which $K(t)$ moves as t varies from 0 to 1.

We conclude this section by mentioning certain examples of simplicial complexes covered by the theorem. The first is the case already mentioned, when $X(\alpha; u, v)$ denotes a *Bézier polynomial* or *rational function* of degree n in the variables (u,v), with α representing the coordinates of the control points or the weights. For n-degree Bézier polynomial functions we may write $\alpha = (\ldots, \boldsymbol{R}_{ijk}, \ldots)$ and $X(\alpha; u, v) = \sum_{ijk} \boldsymbol{R}_{ijk} B_{ijk}(u,v)$ where (u,v) are the coordinates for a point in $\mathcal{T}$, $\boldsymbol{R}_{ijk}$ are the control points, $i+j+k=n$, and B_{ijk} are the Bernstein-Bézier polynomials. In this case $\Omega = R^m = \{\alpha = (\ldots, \boldsymbol{R}_{ijk}, \ldots)\}$. Further, it is clear that the mapping of $(\alpha; u, v)$ into $X(\alpha; u, v)$ is in $C^\infty(\Omega \times \mathcal{T})$. For rational n-degree Bézier surfaces we have $\alpha = (\ldots, \boldsymbol{R}_{ijk}, \ldots, w_{ijk}, \ldots)$ and $X(\alpha; u, v) = \sum_{ijk} \boldsymbol{R}_{ijk} B_{ijk}(u,v) / \sum_{ijk} w_{ijk} B_{ijk}(u,v)$, with w_{ijk}, $i+j+k=n$, denoting the weights. Then we take

$$\Omega = \{\alpha = (\ldots, \boldsymbol{R}_{ijk}, \ldots, w_{ijk}, \ldots) : \textstyle\sum_{ijk} w_{ijk} B_{ijk}(u,v) \neq 0 \text{ for all } (u,v) \in \mathcal{T}\ \},$$

or more simply,

$$\Omega = \{\alpha = (\ldots, \boldsymbol{R}_{ijk}, \ldots, w_{ijk}, \ldots) : w_{ijk} > 0 \text{ for all } i,j,k\ \}.$$

Again, the mapping is in $C^\infty(\Omega \times \mathcal{T})$. Assumption 1 can be satisfied, for example, by letting the control points and the weights be linear functions of t, and we may let the control points for common boundary curves coincide. The real issues [13] are the verification that the mappings $X_i(t)$ are one-to-one with non-singular Jacobians and that the patches are nontangential for all t.

Similar considerations apply in the context of tensor-product spline surfaces and other surface-modeling schemes.[19,20,21,22]

4. Proof of Theorem

In the following, Subsection 4.1 provides some auxiliary results needed to prove the main lemma. The main lemma and its proof appear in Subsection 4.2, and the proof of the theorem itself (Theorem 1) appears in Section 4.3.

4.1. Auxiliary Results

We begin by formulating some technical lemmas needed in the proof of the main lemma.

Lemma 1 *The Jacobians $A_i(t)$ are uniformly non-singular and bounded in norm, i.e., there exist positive constants k_1 and k_2 such that*

$$k_1\|e\| \leq \|A_i(t;P)e\| \leq k_2\|e\|$$

for all vectors $e \in R^2$, all $t \in [0,1]$, all i and all $P \in \mathcal{T}$.
The proof follows by a continuity and compactness argument, using Assumptions 1 and 2.

Lemma 2 *There exist positive constants k_3 and k_4 such that*

$$k_3\|P_1 - P_0\| \leq \|X_i(t;P_1) - X_i(t;P_0)\| \leq k_4\|P_1 - P_0\|$$

for all P_1, $P_0 \in \mathcal{T}$, all i and all $t \in [0,1]$.

Proof. Since the second derivatives with respect to P of $X_i(t;P)$ are bounded uniformly in $[0,1] \times \mathcal{T}$, there exists, by Taylor's theorem, a constant c_1 such that

$$\|X_i(t;P_1) - X_i(t;P_0) - A_i(t;P_0)(P_1 - P_0)\| \leq c_1\|P_1 - P_1\|^2$$

for all P_0, P_1 and all t. Therefore

$$\|X_i(t;P_1) - X_i(t;P_0)\| \geq \|A_i(t;P_0)(P_1 - P_0)\| - c_1\|P_1 - P_0\|^2.$$

By Lemma 1 we then get

$$\|X_i(t;P_1) - X_i(t;P_0)\| \geq k_1\|P_1 - P_0\| - c_1\|P_1 - P_0\|^2.$$

It follows that

$$\|P_1 - P_0\| \leq k_1/(2c_1) \Longrightarrow \|X_i(t;P_1) - X_i(t;P_0)\| \geq (k_1/2)\|P_1 - P_0\|.$$

Further, since the functions $\|X_i(t;P_1) - X_i(t;P_0)\|$ are strictly positive and continuous on the compact set

$$\{(t, P_1, P_0) : t \in [0,1], P_1, P_0 \in \mathcal{T}, \|P_1 - P_0\| \geq k_1/(2c_1)\} \subset R^5$$

we conclude that there exists a constant $c_2 > 0$ so that

$$\|X_i(t;P_1) - X_i(t;P_0)\| \geq c_2 \geq c_2\|P_1 - P_0\|/\mathrm{diam}(\mathcal{T})$$

for all P_1 and P_0 with $\|P_1 - P_0\| \geq k_1/(2c_1)$ and for all t. Here $\mathrm{diam}(\mathcal{T})$ denotes the diameter of the triangle $\mathcal{T}$. Now, taking $k_3 = \min\{c_2/\mathrm{diam}(\mathcal{T}), k_1/2\}$, the first inequality in the statement of the lemma is satisfied.

The second inequality follows from the mean value theorem. In fact, one may take $k_4 = k_2$, based upon standard arguments.[23] □

Lemma 3 *If $S_i(t)$ and $S_j(t)$ are non-adjacent (i.e., disjoint) patches, then there exists a $\delta_1 > 0$ such that*

$$\|X_i(t;P_1) - X_j(t;P_0)\| \geq \delta_1$$

for all t and all P_1, $P_0 \in \mathcal{T}$.

The proof follows immediately by continuity and compactness.

Next consider two adjacent patches $S_i(t)$ and $S_j(t)$. We introduce the notation $\Gamma_i = X_i^{-1}(S_i(t) \cap S_j(t)) \subset \partial\mathcal{T}$ and $\Gamma_j = X_j^{-1}(S_i(t) \cap S_j(t)) \subset \partial\mathcal{T}$ for the inverse images in parameter space of the intersection. By Assumption 3, Γ_i and Γ_j are either both edges or both vertex points in $\mathcal{T}$ and independent of t. For the case when they are edges, let g_i and g_j be unit vectors parallel to Γ_i and Γ_j respectively.

Lemma 4 *Consider first the case when Γ_i and Γ_j are edges. Then if $\delta_2 \in (0,1)$ is given, there exists a constant $\epsilon_1 > 0$ with the following property. For any point $X = X_i(t;P_i) = X_j(t;P_j) \in S_i(t) \cap S_j(t)$, any vectors e_j, e_j pointing from P_i and P_j respectively into $\mathcal{T}$ or along the boundary of $\mathcal{T}$ and satisfying the inequality*

$$|e_i \cdot g_i| + |e_j \cdot g_j| \leq \delta_2\{\|e_i\| + \|e_j\|\} \tag{1}$$

and for all $t \in [0,1]$, the following inequality is valid:

$$\|A_i(t;P_i)e_i - A_j(t;P_j)e_j\| \geq \epsilon_1 \left\{\|A_i(t;P_i)e_i\| + \|A_j(t;P_j)e_j\|\right\}. \tag{2}$$

Next consider the case when Γ_i and Γ_j are both vertex points, $\Gamma_i = \{P_i\}$, and $\Gamma_j = \{P_j\}$ in $\mathcal{T}$. Then for some constant $\epsilon_1 > 0$ the inequality (2) is valid for all vectors e_i and e_j pointing from Γ_i and Γ_j into $\mathcal{T}$.

Proof. By homogeneity, it suffices to prove the lemma when $\|e_i\| + \|e_j\| = 1$.

Assume first that Γ_j and Γ_j are edges and that the statement of the lemma is false. By Lemma 1 the factor $\|A_i(t;P_i)e_i\| + \|A_j(t;P_j)e_j\|$ in the right side of equation (2) is bounded. We then conclude that there exist sequences $t_n \in [0,1]$, e_{in} and $e_{jn} \in R^2$, satisfying (1) and with $\|e_{in}\| + \|e_{jn}\| = 1$ and $P_{in} \in \Gamma_i$, $P_{jn} \in \Gamma_j$ such that $\|A_i(t_n;P_{in})e_{in} - A_j(t_n;P_{jn})e_{jn}\| \to 0$ as $n \to \infty$. By selecting subsequences and using compactness we may assume that $t_n \to t \in [0,1]$, $P_{in} \to P_i \in \Gamma_i$, $P_{jn} \to P_j \in \Gamma_j$, $e_{in} \to e_i \in R^2$ and $e_{jn} \to e_j \in R^2$, with $\|e_i\| + \|e_j\| = 1$. Since the matrices $A_i(t;P)$ and $A_j(t;P)$ depend continuously on t and P it follows that

$$A_i(t;P_i)e_i - A_j(t;P_j)e_j = 0. \tag{3}$$

By continuity of the inner product and the norm it also follows that the inequality (1) is valid for the limits e_i and e_j. Further, at least one of the vectors e_i and e_j is different from the null-vector and hence, by Lemma 1 and (3), both of them are. By inequality (1) at least one of the vectors e_i and e_j points into the interior of $\mathcal{T}$ and therefore, equations (1) and (3) show that we can construct curves emanating from X with properties contradicting Assumption 2 and Definition 2.

For the case that Γ_i and Γ_j are vertex points the argumentation is similar, using Assumption 2. In this case we do not need the inequality (1) to guarantee that the limits e_i and e_j point into the *interior* of $\mathcal{T}$. The details are omitted. □

The next lemma is a reformulation of the geometric property that two adjacent patches are nontangential with disjoint interiors.

Lemma 5 *There exists a constant* $\kappa > 0$ *such that (for adjacent patches)*

$$\|X_i(t;P_i) - X_j(t;P_j)\| \geq \kappa \min\{\|X_i(t;P_i) - X\| + \|X_j(t;P_j) - X\| : X \in S_i(t) \cap S_j(t)\}$$

for all P_i, $P_j \in \mathcal{T}$, *and all* $t \in [0,1]$.

Proof. Assume that the statement is false. Then there exist sequences $\kappa_n > 0$, $t_n \in [0,1]$, $P_{in} \in \mathcal{T}$ and $P_{jn} \in \mathcal{T}$ such that, as $n \to \infty$, $\kappa_n \to 0$ and

$$\|X_i(t_n;P_{in}) - X_j(t_n;P_{jn})\| \leq \kappa_n \min\{\|X_i(t_n;P_{in}) - X\| + \|X_j(t_n;P_{jn}) - X\| : X \in S_i(t) \cap S_j(t)\}. \quad (4)$$

Since the quantity $\|X_i(t_n;P_{in}) - X\| + \|X_j(t_n;P_{jn}) - X\|$ is bounded, it follows that $X_i(t_n;P_{in}) - X_j(t_n;P_{jn}) \to 0$ as $n \to \infty$.

Selecting convergent subsequences, such that $t_n \to t$, $P_{in} \to P_i$, $P_{jn} \to P_j$ we may conclude that

$$X_i(t;P_i) = X_j(t;P_j) \in S_i(t) \cap S_j(t) \quad \text{and} \quad P_i \in \Gamma_i, P_j \in \Gamma_j.$$

Now first take $Q_{in} \in \Gamma_i$ so that $\|P_{in} - Q_{in}\|$ is minimal and then take $Q_{jn} \in \Gamma_j$ so that $X_i(t_n;Q_{in}) = X_j(t_n;Q_{jn}) \in S_i(t) \cap S_j(t)$. Since $P_{in} \to P_i \in \Gamma_i$, it follows that

$$Q_{in} - P_{in} \to 0 \quad (5)$$

and, by Lemma 2, that $X_i(t_n;P_{in}) - X_i(t_n;Q_{in}) \to 0$ as $n \to \infty$. Then, also,

$$X_j(t_n;Q_{jn}) - X_j(t_n;P_{jn}) = X_i(t_n;Q_{in}) - X_j(t_n;P_{jn}) = (X_i(t_n;Q_{in}) - X_i(t_n;P_{in})) + (X_i(t_n;P_{in}) - X_j(t_n;P_{jn})) \to 0 + 0 = 0.$$

This implies, again by Lemma 2, that

$$Q_{jn} - P_{jn} \to 0 \quad (6)$$

as $n \to \infty$. Further

$$\begin{aligned} X_i(t_n;P_{in}) - X_j(t_n;P_{jn}) = \\ X_i(t_n;P_{in}) - X_i(t_n;Q_{in}) + X_j(t_n;Q_{jn}) - X_j(t_n;P_{jn}) = \\ A_i(t_n;Q_{in})(P_{in} - Q_{in}) - A_j(t_n;Q_{jn})(P_{jn} - Q_{jn}) + \\ (X_i(t_n;P_{in}) - X_i(t_n;Q_{in})) - A_i(t_n;Q_{in})(P_{in} - Q_{in}) + \\ -(X_j(t_n;P_{jn}) - X_j(t_n;Q_{jn})) + A_j(t_n;Q_{jn})(P_{jn} - Q_{jn}). \end{aligned}$$

By Taylor's theorem, we have for some constant c_4,

$$\|X_i(t;P_i) - X_i(t;P) - A_i(t;P)(P_i - P)\| \leq c_4\|P_i - P\|^2$$

for all P_i, $P \in \mathcal{T}$ and for all t and all i. We then get

$$\begin{aligned}&\|X_i(t_n;P_{in}) - X_j(t_n;P_{jn})\| \geq \\ &\quad \|A_i(t_n;Q_{in})(P_{in}-Q_{in}) - A_j(t_n;Q_{jn})(P_{jn}-Q_{jn})\| - \\ &\quad c_4(\|P_{in}-Q_{in}\|^2 + \|P_{jn}-Q_{jn}\|^2).\end{aligned}$$

By the choice of Q_{in} it is clear that the condition (1) of Lemma 4 is satisfied for some $\delta_2 \in (0,1)$, with $e_i = P_{in} - Q_{in}$ and $e_j = P_{jn} - Q_{jn}$. Using also Lemma 1, we conclude that, for some constant $\epsilon_1 > 0$

$$\begin{aligned}&\|X_i(t_n;P_{in}) - X_j(t_n;P_{jn})\| \geq \\ &\quad k_1\epsilon_1(\|P_{in}-Q_{in}\| + \|P_{jn}-Q_{jn}\|) - c_4(\|P_{in}-Q_{in}\|^2 + \|P_{jn}-Q_{jn}\|^2). \quad (7)\end{aligned}$$

Now combining (7) and (4) we have

$$\begin{aligned}&k_1\epsilon_1(\|P_{in}-Q_{in}\| + \|P_{jn}-Q_{jn}\|) - c_4(\|P_{in}-Q_{in}\|^2 + \|P_{jn}-Q_{jn}\|^2) \leq \\ &\quad \kappa_n\{\|X_i(t_n;P_{in}) - X_i(t_n;Q_{in})\| + \|X_j(t_n;P_{jn}) - X_j(t_n;Q_{jn})\|\}.\end{aligned}$$

Finally, using Lemma 2, we conclude that

$$\begin{aligned}&k_1\epsilon_1(\|P_{in}-Q_{in}\| + \|P_{jn}-Q_{jn}\|) - c_4(\|P_{in}-Q_{in}\|^2 + \|P_{jn}-Q_{jn}\|^2) \leq \\ &\quad \kappa_n k_4(\|P_{in}-Q_{in}\| + \|P_{jn}-Q_{jn}\|)\end{aligned}$$

which, by (5) and (6) gives a contradiction as $n \to \infty$. This completes the proof. □

4.2. Main Lemma

We now give the main lemma. To establish the notation, consider the mappings

$$f_t : K(0) \to K(t)$$

defined by Assumptions 1 and 2 and

$$f_t|_{S_i(0)} = X_i(t) \circ X_i^{-1}(0).$$

Similarly, let $f_{t,s}$ denote the mappings

$$f_{t,s} : K(s) \to K(t) \subset R^3$$

given by $f_{t,s} = f_t \circ f_s^{-1}$ or, equivalently $f_{t,s}|_{S_i(s)} = X_i(t) \circ X_i^{-1}(s)$. We have

$$f_t = f_{t,0}, \quad f_{t,t} = Id_{K(t)}, \quad \text{and} \quad f_{t,r} = f_{t,s} \circ f_{s,r}.$$

Similarly, let $l_{t,s} : K(s) \to R^3$ be defined by the mapping of p into $f_{t,s}(p) - p = l_{t,s}(p) \in R^3$, *i.e.*, so that

$$f_{t,s} = Id_{K(s)} + l_{t,s}.$$

Further, if $p = p(0) \in K(0)$ is given, let $p(t) = f_t(p)$, and, similarly, $q(t) = f_t(q)$. Clearly

$$l_{t,s}(p(s)) = p(t) - p(s).$$

Lemma 6 *For every $\beta > 0$ there exists a $\delta > 0$ such that*

$$\|l_{t,s}(p(s)) - l_{t,s}(q(s))\| \leq \beta \|p(s) - q(s)\|$$

whenever $p(s)$ and $q(s) \in K(s)$ and $|t - s| < \delta$, i.e., so that $l_{t,s}$ is a Lipschitz mapping with constant β.

Proof. The proof is divided into three cases.

a. $p(s)$ and $q(s)$ in the same patch: $p(s) \in S_i(s)$, $q(s) \in S_i(s)$.

b. $p(s)$ and $q(s)$ in adjacent patches: $p(s) \in S_i(s)$, $q(s) \in S_j(s)$, $S_i(s)$ and $S_j(s)$ are adjacent.

c. $p(s)$ and $q(s)$ in disjoint patches: $p(s) \in S_i(s)$, $q(s) \in S_j(s)$, $S_i(s) \cap S_j(s) = \emptyset$.

Case a: With the previous notation we may write, for some i,

$$p(s) = X_i(s; P_1), \quad q(s) = X_i(s; P_0),$$

$$p(t) = X_i(t; P_1), \quad q(t) = X_i(t; P_0),$$

with P_1 and $P_0 \in \mathcal{T}$. Then

$$\begin{gathered} l_{t,s}(p(s)) - l_{t,s}(q(s)) = \\ p(t) - p(s) - (q(t) - q(s)) = \\ X_i(t; P_1) - X_i(s; P_1) - (X_i(t; P_0) - X_i(s; P_0)) = \\ X_i(t; P_1) - X_i(t; P_0) - (X_i(s; P_1) - X_i(s; P_0)) = \\ G(t, P_0, P_1) - G(s, P_0, P_1) \end{gathered}$$

where we have introduced the vector-valued function

$$G(t, P_0, P_1) = X_i(t; P_1) - X_i(t; P_0).$$

Now,[23]

$$\|G(t, P_0, P_1) - G(s, P_0, P_1)\| \leq |t - s| \sup \left\{ \|\dot{G}(\theta, P_0, P_1)\| : s < \theta < t \right\}$$

where the dot denotes differentiation with respect to t. Further, by Assumption 1 and the fact that $X_{i,P}$ is in $C^2(\Omega \times \mathcal{T})$ it follows that the mixed derivatives $\dot{X}_{i,P} = \dot{A}_i$ are in $C([0,1] \times \mathcal{T})$ and hence bounded. Since $\dot{G}(t, P_0, P_1) = \dot{X}_i(t; P_1) - \dot{X}_i(t; P_0)$, we conclude [23] that there exists a constant c_4 so that

$$\begin{gathered} \|\dot{G}(\theta, P_1, P_0)\| = \|\dot{X}_i(\theta; P_1) - \dot{X}_i(\theta; P_0)\| \leq \\ \sup\{\|\dot{A}_i(\theta; (1-r)P_0 + rP_1)\| : 0 \leq r \leq 1\} \|P_1 - P_0\| \leq c_4 \|P_1 - P_0\|. \end{gathered}$$

Consequently

$$\|l_{t,s}(p(s)) - l_{t,s}(q(s))\| \leq c_4 |t - s| \cdot \|P_1 - P_0\|$$

for all t, s, P_0 and P_1. By Lemma 2, there exists $k_3 > 0$ such that

$$k_3\|P_1 - P_0\| \le \|X_i(s; P_1) - X_i(s; P_0)\| = \|p(s) - q(s)\|.$$

Therefore,

$$\|l_{t,s}(p(s)) - l_{t,s}(q(s))\| \le c_4|t - s| \cdot \|p(s) - q(s)\|/k_3.$$

Taking $\delta = \beta k_3/c_4$ completes the argument for Case a.

Case b: Now we may write

$$p(s) = X_i(s; P_i), \quad q(s) = X_j(s; P_j),$$

$$p(t) = X_i(t; P_i), \quad q(t) = X_j(t; P_j),$$

with P_i and $P_j \in \mathcal{T}$.

Now, for a fixed s, let $r = r(s)$ be the point in $S_i(s) \cap S_j(s)$ which minimizes the expression $\|p(s) - r\| + \|q(s) - r\|$ (cf. Lemma 5). Let $r(t) = l_{t,s}(r) = l_{t,s}(r(s))$. Then arguing as in the previous case we obtain

$$\begin{aligned} &\|l_{t,s}(p(s)) - l_{t,s}(q(s))\| \le \\ &\quad \|l_{t,s}(p(s)) - l_{t,s}(r(s))\| + \|l_{t,s}(q(s)) - l_{t,s}(r(s))\| \le \\ &\quad c_4|t - s|\{\|p(s) - r(s)\| + \|q(s) - r(s)\|\}/k_3. \end{aligned}$$

Then, by Lemma 5, we get

$$\|l_{t,s}(p(s)) - l_{t,s}(q(s))\| \le (c_4|t - s|/(k_3\kappa))\|p(s) - q(s)\|$$

and taking $\delta = \beta\kappa k_3/c_4$ completes the argument for Case b.

Case c: We have, with notation as in Case b,

$$\begin{gathered} l_{t,s}(p(s)) - l_{t,s}(q(s)) = \\ X_i(t; P_i) - X_i(s; P_i) - (X_j(t; P_j) - X_j(s; P_j)). \end{gathered}$$

Here, the right hand side is a continuous function of $(t, s, P_i, P_j) \in [0,1]^2 \times \mathcal{T}^2$, which is zero when $s = t$. By uniform continuity, given any $\epsilon > 0$, there exists $\delta > 0$ such that

$$|t - s| < \delta \Longrightarrow \|l_{t,s}(p(s)) - l_{t,s}(q(s))\| < \epsilon.$$

By Lemma 3, $\|p(s) - q(s)\| \ge \delta_1$. Taking $\epsilon = \beta\delta_1$ we conclude that

$$|t - s| < \delta \Longrightarrow \|l_{t,s}(p(s)) - l_{t,s}(q(s))\| \le \beta\|p(s) - q(s)\|.$$

Finally, choosing δ as the minimal value obtained in the Cases a, b and c and for all combinations of i and j, completes the proof of Lemma 6. □

4.3. Proof of Theorem 1

It suffices to prove that $K(1)$ is linked by an ambient isotopy to $K(0)$. Let $\beta \in (0,1)$ be given. Consider a subdivision $0 = t_0 < t_1 < \ldots < t_i < t_{i+1} < \ldots < t_M = 1$ of the interval $[0,1]$. We introduce the shorter notation

$$l_i = l_{t_{i+1},t_i}, \quad f_i = f_{t_{i+1},t_i}, \quad \text{and} \quad f = f_{1,0}.$$

Then $f_i = Id_{K(t_i)} + l_i$, $f = f_{M-1} \circ f_{M-2} \circ \ldots \circ f_i \circ \ldots \circ f_0$ and $f_i : K(t_i) \to K(t_{i+1})$ is a homeomorphism, as well as $f : K(0) \to K(1)$. By Lemma 6 we may choose the subdivision so that

$$\|l_i(p) - l_i(q)\| \leq (\beta/3)\|p - q\|$$

for all p and $q \in K(t_i)$, $1 \leq i \leq M$.

We invoke the following Extension Lemma, due to Whitney [25] and McShane.[26] We also indicate the proof, which is elementary.

Theorem 2 *Let K be an arbitrary subset of a normed linear space B. Further, let $l : K \to R$ be a Lipschitz mapping with constant γ, so that,*

$$|l(x) - l(y)| \leq \gamma\|x - y\|$$

for all x and $y \in K$. Then there exists an extension $\tilde{l} : B \to R$ so that

$$|\tilde{l}(x) - \tilde{l}(y)| \leq \gamma\|x - y\|$$

for all x and $y \in B$ and so that $\tilde{l}(x) = l(x)$ for all $x \in K$.

Proof. It is straightforward to verify that the choice

$$\tilde{l}(x) = \inf\{l(z) + \gamma\|x - z\| : z \in K\}$$

or

$$\tilde{l}(x) = \sup\{l(z) - \gamma\|x - z\| : z \in K\}$$

will do. □

Applying this lemma to each component of l_i, with $\gamma = \beta/3$, we conclude that there exist extensions $\tilde{l}_i : R^3 \to R^3$ such that

$$\|\tilde{l}_i(p) - \tilde{l}_i(q)\| \leq \beta\|p - q\|$$

for all p and $q \in R^3$, and so that $\tilde{l}_i|_{K(t_i)} = l_i$.

Proof of Theorem 1:

For $t = t_i$, $1 \leq i \leq M$, take $\tilde{f}_i = Id_{R^3} + \tilde{l}_i$, and for $t \in (t_{i-1}, t_i]$, define

$$\tilde{f}_i(t) = Id_{R^3} + \frac{t - t_{i-1}}{t_i - t_{i-1}}\tilde{l}_i.$$

Since $\beta < 1$ and $\frac{t-t_{i-1}}{t_i-t_{i-1}} \leq 1$, $\tilde{f}_i(t) - Id_{R^3}$ is a contraction mapping, and we may apply the contraction-mapping principle [27] to conclude that $\tilde{f}_i(t)$ is a homeomorphism from R^3 to R^3 for $t \in (t_{i-1}, t_i]$. Here, the contraction-mapping principle states that given a mapping $T : R^3 \to R^3$ with $\|T(x) - T(y)\| \leq \kappa \|x - y\|$ for some $\kappa \in [0, 1)$ and for all x and $y \in R^3$, then there exists a unique $x_0 \in R^3$ such that $T(x_0) = x_0$; this result can be easily applied in our case to show that $\tilde{f}_i(t)$ has a continuous inverse. Now, for $t \in (t_{i-1}, t_i]$, define

$$F(t) = \tilde{f}_i \circ \tilde{f}_{i-1} \circ \tilde{f}_{i-2} \circ \ldots \circ \tilde{f}_1(t)$$

and $F(0) = Id_{R^3}$. It is clear that $F(t)$, $0 \leq t \leq 1$, gives an ambient isotopy between $K(0)$ and $K(1)$. □

In order to prove Corollary 1, we need only redefine the previous mappings $f_{t,s}$ and $l_{t,s}$ so that they are equal to the identity and the null-function respectively on the complement of $\mathcal{O}$, and make a corresponding modification of the proof of Lemma 6.

5. Conclusion

In this paper we have provided sufficient conditions that a perturbation of a given object, which is the finite union of properly-joined two-dimensional curvilinear simplexes in R^3, retains its topological form, *i.e.*, that there exists an ambient isotopy linking the original and perturbed objects. The conditions essentially require that the original object should be continuously perturbed into its new position in such a way that no self-intersections or extraneous intersections are introduced, and in particular, adjacent patches remain nontangential.

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