Cyclically Ordered Faces

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Brian,

You posed the question, "Can a set of faces with a common vertex v be cyclically ordered?", wherein you noted that this is easily done in the plane and wondered whether it was possible to do so in higher dimensions. I think that the following fully answers your question, with the answer being independent of the dimension of the model, but depending only upon the local neighborhood (nbhd) structure at v.

An easy answer is, of course, this can be trivially done. Namely just pick any one face, assign it the number 1, pick any other face, assign it the number two, ..., continuing in this manner of picking distinct faces until all have been exhausted. However, this was not the intent. There is more desired with respect to the cyclic ordering. Namely, what is implied is the following:

1. each face has two edges incident to v,
2. if a particular face has number n, and it shares an edge with another face, then it should be possible to traverse consistently (either clockwise or counterclockwise) around v, so that the face numbered $n+1$ shares an edge with the face indexed by n,
3. following the numbered faces from $n-1$ to n to $n+1$, should require traversal over the two edges, each having vertex v, of the face indexed by n.

Now consider the two kinds of topological connectivity typically used in geometric modeling, manifold models and non-manifold models. If we restrict attention to those two classes, we eliminate esoteric counterexamples, like the Cantor set, resulting in the following theorem:

Theorem: For the class of manifold and non-manifold models, given a vertex v, there will exist a cyclic ordering of the faces incident to v, having properties 1, 2 & 3, listed above if and only if v has a nbhd which is a 2-manifold.

Proof: (if) In this case, some $\epsilon > 0$ can be found such that the nbhd H about v given by

$$H = B_\epsilon(v) \cap star(v)$$

can be considered to be planar and homeomorphic to the interior of a disc in the plane; where $B_\epsilon(v)$ is the open 3-ball of radius ϵ about v and $star(v)$ is the set of all edges and faces incident to v.

Then $frontier(H)$ is homeomorphic to C_ϵ, denoting the circle of radius ϵ about v. Observe that $frontier(H) \cap edges(v)$ will be a set of isolated points, where $edges(v)$ is the set of all edges incident at v. Each point in this set has a corresponding unique homeomorphic image in C_ϵ, where these images can be cyclically ordered by picking an initial point and then ordering the others consecutively as they are encountered within a consistently oriented traversal of C_ϵ. Then assign this same ordering to the corresponding pre-images and index each face by the lower of the two values from these pre-images.

(only if) (by contrapositive) If v does not have a 2-manifold nbhd, then its nbhds must be non-manifold. So, there must be an edge with more than 2 shared faces. (We ignore here the trivial cases of any 2 shared faces being non-manifold solely because of an additional shared edge as being non-germane.) However, it should be clear that for any possible assignment of n and $n+1$ to any two of the faces, it is then impossible to assign the third face either $n-1$ or $n+2$ to still satisfy #2 and #3, above. A simple example will make this clear. Consider the origin in R^3, with one face being the triangle bounded by the three points

$$(0,0,0),(1,0,0),(1,1,0),$$

another face being the triangle bounded by the three points

$$(0,0,0),(1,0,0),(1,-1,0),$$

and another face being the triangle bounded by the three points

$$(0,0,0),(1,0,0),(1,0,1).$$

Without loss of generality, assign 1 and 2 to the first two faces listed. Then, the remaining face shares neither the edge between $(0,0,0)$ and $(1,1,0)$ nor the edge between $(0,0,0)$ and $(1,-1,0)$, thereby failing to meet #3. Alternatively, if the first face is assigned 1 and the last face is assigned 2, then there is no way to assign the third value without contradicting the intent of #3.

Corollary: If the model is a manifold, then this ordering can be done for each vertex of the model.

Proof: The criterion of the theorem was local for a particular vertex, v. If the object is a manifold, then all vertices have the required property. However, note that the theorem permits this ordering for even those vertices in non-manifold models, which satisfy the proper local criteria.