The Diversity of Topological Applications within Computer Aided Geometric Design

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Abstract

Boolean algebras of regular closed sets, continuous functions, tame homeomorphisms, and Betti numbers are representative of the spectrum of topological tools that have been useful within computer aided geometric design (CAGD). The history of mathematics is rich with examples where the investigation of applications leads to extensions of existing theory. In that spirit, representative applications of topology to CAGD are presented. The intent is to present these examples in mathematical language within their larger mathematical context, so that other topologists might be encouraged to simultaneously enrich CAGD practice and mathematical theory. The authors' own research has benefited from such synergy, in that preparation of this article has resulted in new findings, which are presented herein.

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0. Introduction

The intent of this article is to demonstrate the diversity of topological techniques [4] that have been employed within computer aided geometric design (CAGD). Space constraints prohibit detailed exposition of this rich diversity. Rather, this article is intended to whet the reader's appetite for further investigation. Although there are never clear boundaries in a discipline, for the purposes of this paper, four diverse topological subfields, each paired with a topical CAGD example, will be emphasized:

1. *set-theoretic topology*: the Boolean algebra of regular closed sets applied to the theoretical foundations of solid modeling,
2. *point-set topology*: continuous functions on metric spaces as a model for the transformation from design to manufacturing,
3. *geometric topology*: tame homeomorphisms for emerging tolerance theory,
4. *algebraic topology*: Betti numbers for the validation of specific CAGD models.

Topologists have made significant contributions to computer science. Testimony of that influence within theoretical computer science is offered by the monograph *"Topology and Category Theory in Computer Science"* [13]. Additionally, the relevance of topology to computer graphics has also been demonstrated [10]. In a similar spirit, this article discusses topological opportunities within CAGD and attempts to spark the curiosity of a wide range of topologists.

1. Regular Closed Sets and Related Algebras

A key insight in the theory of Stone-Čech compactifications was the interplay between topological constructs and corresponding algebraic statements. Then, difficult topological problems became accessible via algebraic techniques. The monograph, *"Rings of Continuous Functions"* [5], may be viewed as an extended exposition of that theme. Similarly, computer aided geometric design has evolved from an initial emphasis upon geometric concerns to an intermediate focus upon topological issues to a maturing perspective upon supporting algebras. Within CAGD, topology concentrates upon adjacency relations amongst vertices, edges and faces, where the Euler-Poincaré Equation provides a classical algebraic analogue.

Solid modeling is a prominent speciality within computer aided geometric design. The proliferation of inexpensive graphics workstations promises to allow solid modeling to become the dominant mode of CAGD modeling in the near future. Solid modeling may be considered as a system of operands and operators. The solid operands are required to enclose positive finite volumes and are compact 3-manifolds with boundaries. Trivial examples of such operands are the closed unit cube (given by $[0,1] \times [0,1] \times [0,1]$) and by

[4] Because of the interdisciplinary nature of this paper, original mathematical references will not always be cited. Some citations will be to application articles, which, in turn, can provide reference to the original mathematical sources.

parallelepipeds (which are understood to include both their bounding planar surfaces and their enclosed volumes). Other elementary examples of these 3-manifold operands would have boundary surfaces given by cylinders, tori and 2-spheres.

One technique for building more sophisticated solid models utilizes binary combinatorial operators upon the solid operands. Initially, the solid operands were conceived primarily as point sets, and the necessary binary operators were seen as set union, intersection and subtraction. However, the intent was that the resultant of any such binary operation would be a valid solid and, thus, could be considered as an operand for another binary operation. The set intersection operation formed the basic kernel of algorithms for all three operations and it was readily apparent that the set theoretic intersection operator could easily yield resultants which were not valid solids [26] (e.g. consider two closed unit cubes aligned so that their intersection is merely a unit square). The Boolean algebra of regular closed sets was seen as a formal expression of the necessary binary operators [26].

The intent of *"intersection"* was replaced by the Boolean meet. The intent of *"union"* was replaced by the Boolean join. If A and B are solids, the intent of the *"subtraction"* of B from A, can be specified notationally as

$$A - B = cl_X int_X (A \cap B'), \tag{1}$$

where B' indicates the set complementation of B. These three operators are collectively referred to as *"Booleans"*. However, the original terminology persists, and these three operators are referred to individually, albeit somewhat imprecisely, as union, intersection and subtraction.

If *"Rings of Continuous Functions"* can be seen as further investigation of the original topological/algebraic interplay expressed within the Stone-Čech compactification, then, in a similar manner, the recent doctoral dissertation of V. Shapiro [19], may be seen as an extended exploration of the topological/algebraic interplay within the broad realm of geometric modeling. Geometric modeling also includes other non-solid operand/operator schemes. V. Shapiro [19] seeks a unified algebraic formalism for all such CAGD modeling schemes. V. Shapiro is specifically motivated in his search by his realization that in current CAGD "... the lack of a common formal language leads to the proliferation of informal concepts and results in many redundant efforts that are often ill-defined." [19].

V. Shapiro suggests that "...the key to development of such a language is an algebra using the set operations together with operations of closure and connected component." [20]. As a specialization of the standard Boolean algebra, he investigates [19] closure algebras and their generation by a finite number of polynomial primitives. In turn, the relation between these polynomial primitives and a decomposition of R^n is found to be not well understood, and, as an alternative, "Whitney regular stratifications" are pursued. This path culminates in an investigation of the Brouwerian algebra of closed (or open) subsets of R^n, "... that also contains the Boolean (sub)algebra of closed regular sets." [20].

For solid modeling, the space over which a Boolean algebra is considered is R^3. The concern over retaining bounded volumes within R^3 motivated the further restriction that all solids must also be semi-analytic sets, resulting in further specialization of the algebra. The knowledgeable reader is aware that regular closed sets need not be restricted to R^3.

In fact, CAGD applications within R and R^2 have already proven quite useful. The need for modeling in R^n, where n is a fixed, but arbitrary positive integer, is gaining increasing importance. However, several of the assumptions made about valid resultants enclosing finite volumes restricted the initial versions of solid modeling code from being gracefully integrated with software for modeling lower and higher dimensional geometry.

The seminal work on providing generalized data structures, algorithms, and a software engineering basis [5] for that integration was performed by K. J. Weiler [28, 29] [6] under his terminology of *"non-manifold topology"* [7]. While recognizing the intimate relationship between CAGD topological constructs and the underlying computational geometry, the work of K. J. Weiler [28, 29, 30] was a catalyst for computational topology to achieve a mature software engineering role as an independent subsystem. He developed geometry and topology subsystems independently, explaining his motivation as follows [29]:

Use of topological properties can simplify modeling algorithms and greatly improve their efficiency. However, for several reasons, topology can be even more useful when it serves as the framework around which the geometric modeling representation can be built. First, once the topological and geometric domain which the representation is intended to cover has been defined, and the corresponding topological representation has been selected, the topological portion of the *implementation* generally is less subject to change than the geometric portion of the implementation. Second, separation of topological and geometric information in a geometric modeling representation provides a more systematic approach to implementation, providing for simpler creation, verification, and analysis of the model.

The major legacy of this work has been "...to eliminate the formerly large conceptual gaps between the different dimensional element modeling techniques and provide a uniform environment for these techniques in both a conceptual and practical sense." [30]. Previously, within the CAGD community, the pervasive thinking required sharp boundaries separating curve, surface and solid modeling techniques from each other. CAGD researchers "...concentrated on ... [these] differences rather than their unification." [30]. It was K. J. Weiler's observation that human designers "...use a continuum of related geometric entities in a single domain rather than separate geometric entity sets in separate domains." [30]. Then, "...the relationship between these different dimensional entities, rather than their differences..." [30] became the focal point. Unification, rather than differentiation, was then pursued. This quest for formal unification is likely to be aided by such algebraic pursuits as V. Shapiro's [19].

[5] While software engineering issues are *not* the focus of this article, it is worthwhile to note that K. J. Weiler utilizes an edge-based representation, where each edge serves as a reference from which adjacency relations are determined. Other practitioners [7] prefer a vertex based approach.

[6] The integration of surface and solid modeling techniques was also undertaken independently in the early 1980's at the University of Utah [1].

[7] Recent discussions within the CAGD community [8] have raised some concerns about the lack of precision in the use of the term "non-manifold topology" and the expression "mixed-dimensional modeling" has been advocated as more suggestive.

2. Functions from a Design Space to a Manufacturing Space

The role of topological neighborhoods to discriminate between candidate designs was initiated by H. Yoshikawa [31] and was refined by T. Tomiyama and H. Yoshikawa [27] and by T. Taura and H. Yoshikawa [25]. In our opinion, this insight that topological spaces could be used to model the conceptual design process is a significant contribution to design theory. These papers were written by practitioners, not topologists, and were intended for an audience of practitioners. As such, it is not surprising to find some mathematical errors. Even so, their underlying insight provided the stimulus for D. W. Rosen and T. J. Peters [18] to investigate the role of continuous functions in modeling the transformation of a design into its manufactured realization.

Within the specific design domain of thin-walled components, data representations were created for designs. A metric topology was imposed upon this set of designs [18]. Within this domain, it was well known how to convert a design into an appropriate manufacturing representation. A topology was imposed upon the set of manufacturing representations and a function was defined which mapped each design into its corresponding manufacturing representation. It was shown that this function need not be continuous, thereby offering a formal basis for the mechanical engineering folklore that a small design change can significantly increase product cost via the necessitated manufacturing modifications.

While the details [18] of the metric definition and the manufacturing topology are too extensive to be expressed here, an illustrative example of the function discontinuity will be presented. Briefly summarizing, if two components differ only in the numeric values assigned to their parameters (e.g., depth, width, diameter, etc.) then, for any two such components c_1, c_2, their distance, $d(c_1, c_2)$ is given by a normalized sum of the absolute values of the differences between corresponding parameter values [8], where the normalization returns a value strictly less than one [9]. Otherwise, the distance between any two components not having this parameter relationship is arbitrarily assigned the value of one. The critical relevant aspect of the manufacturing topology is its definition by a finite basis of open sets. Current manufacturing practice usually assigns all designs into one of a few well defined manufacturing processes, leading to a basis of only finitely many open sets for the manufacturing space. This is primarily to minimize the expensive tooling costs attendant with each distinct manufacturing process, suggesting the dependency of the manufacturing space topology upon some cost factors.

Consider the simple component shown in Figure 1 (a flat wall with a cylindrical boss). When the boss is vertical it can be manufactured very easily via a tool that can be vertically extracted along the exterior cylindrical boundaries. However, if the boss is tilted from this vertical orientation, then the tool can no longer be vertically extracted and more sophisticated manufacturing processes must be utilized.

In this example, the metric would measure the difference in the values of these ori-

[8] This definition is equivalent to a bounded taxi-cab metric in R^n, for an appropriate choice of n.

[9] This normalization is *not* essential for the underlying mathematics, but merely is more compatible with engineers' intuitive notions regarding the design space relative to a particular design/manufacturing domain.

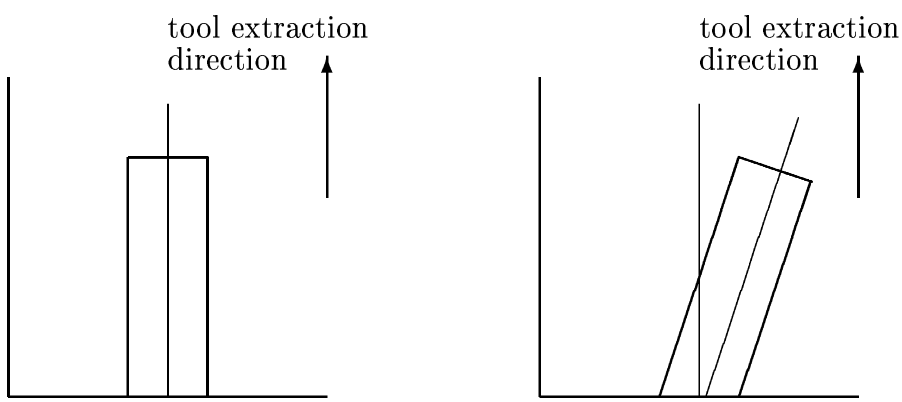

Figure 1: Tool Extraction Difficulty When Boss Is Not Vertical.

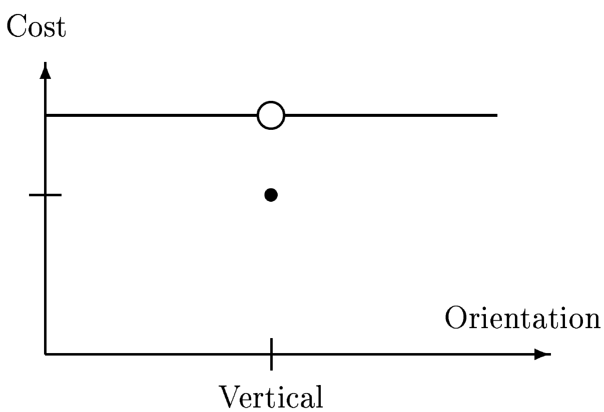

Figure 2: Graph of Cost Versus Orientation.

entation angles. Thus, the function mapping design representation to cost has a point discontinuity about this vertical orientation, as shown in the graph of tooling cost versus tilt angle, given in Figure 2. The subtle shift to considering the image as a cost space is permissible because the topology imposed upon the manufacturing space is a weak topology generated from the topology upon the cost space, where the functional mappings from the manufacturing space to the cost space are well understood.

While the example illustrates a particular discontinuity, a more global view also lends some new insight into the underlying issues. Since each parameter is defined on an interval of the form $(0, k]$, for an appropriately chosen k, it should be clear that this metric topology upon the design space is locally connected [10], whereas the topology on the manufacturing space is totally disconnected [11]. This is significant, because it is then possible to show

[10] The collection of all parameters can be viewed as a product space, where each factor is a connected sub-interval of the real line.

[11] This finite topology is also zero-dimensional, but this seems to be of little mathematical interest.

that the specified function could not be continuous because of a local failure to preserve connectivity. This begins to partially address the nagging question as to whether the specific display of discontinuity at a particular point could be eliminated by choice of another topology, *which would still appropriately model the engineering concerns*. The answer appears to lie in critical differences between the intuitive views of the engineering demands upon both spaces.

In the design space, a metric is desirable so as to be able to quantitatively measure the difference between two designs. Furthermore, since two designs may differ (as in Figure 1) only by numerical variances in the values of specific parameters, it appears that local connectivity is a modest additional requirement to impose upon the design space. Such a locally connected domain and the totally disconnected range preclude continuity when the transformation function maps two designs, which differ only in their parameter values, into disjoint open sets in the manufacturing space.

While such topological incompatibilities preclude continuity, another question remains. Is it possible to define other topologies that would support continuous design-manufacturing transformation functions? For instance, first consider the transformation function as a mapping purely between the underlying sets, with no topologies yet imposed. Then define the totally disconnected topology upon the manufacturing space, as previously indicated. Consider the design space in the weak topology specified by defining open sets to be the inverse images of open sets of the manufacturing space. While this necessarily guarantees continuity, this implies full knowledge of the manufacturing space before the design is finalized, a situation which rarely occurs in practice.

However, emerging manufacturing processes (such as stereolithography and laser machining, where there is a fixed tooling cost for a wide range of manufacturing possibilities, as opposed to the traditional partitioning of manufacturing space based upon a finite number of distinct tools) could relax the assumed restrictions relative to a totally disconnected manufacturing space, whereby continuous transformations functions may result. The critical point is *not* that this or any particular model is definitive, but that formal analyses of the design spaces via topological concepts may provide valuable understanding of these difficult manufacturability evaluations. Furthermore, the role of the weak topology as a model for preserving continuity supports our conjecture that the incorporation of manufacturing knowledge into design activity, as is desirable in modern engineering practice, can result in significant benefits within the design/manufacturing transition, but, correspondingly requires knowledge of the manufacturing space at design time, thereby implying fundamental modifications to many existing design paradigms.

3. Geometric Topology to Preserve Form in Tolerance

An example of the application of geometric topology to tolerance theory within CAGD is given. The formal basis for this theory relies upon geometric topology pioneered by R. H. Bing [2] and his geometric topology contemporaries. Currently, such rigorous tolerance theory is restricted to a very narrowly defined subclass of polyhedra.

Tolerance theory informally assumes that two objects within tolerance of each other must have the *'same form'*, suggesting some undefined notion of topological equivalence. Past engineering practice has not necessitated a formalization of this *'same form'* notion, since two human experts could agree when the condition failed to be met. The primary question is, "If the tolerance parameters on a set S are exercised, creating a new set S', then must S and S' have the *'same form'*?" The use of geometric topology permits a precise, consistent, verifiable mathematical definition for this *'same form'* condition. Appropriate mathematical formalism is necessary for inclusion of tolerance modeling and analysis in modern CAGD systems, as well as in their supporting computer aided manufacturing systems.

The issue of tolerance within CAGD was addressed as long ago as 1983 by the prominent CAGD pioneer, A. A. G. Requicha [14]. More recently, he [15] stated that formal tolerance theory was still in its infancy. In a recent survey [9], N. P. Juster stresses that the success of modern manufacturing industries is dependent upon computer modeling of tolerance. N. P. Juster cites some modest advances, but emphasizes the immaturity of the field. Unfortunately, this Juster survey appears to have been prepared in advance of significant new developments by M. Boyer and N. F. Stewart [3, 4] and by N. F. Stewart [22]. N. F. Stewart's work was cited by A. A. G. Requicha [16] as a critical new development at a major CAGD conference, sponsored by the Society of Industrial and Applied Mathematics (SIAM). In a recent survey article, *"Solid Modeling and Beyond"*, [17], M. Boyer and N. F. Stewart are lauded, as follows;

...we see progress in the mathematics of tolerance specifications, which also define sets of sets, called *variational classes*. ... Recent work by Boyer and Stewart at the University of Montreal proposes sharp mathematical characterizations for variational classes in terms of regularity in a topology related to the Hausdorff distance between sets.

The original work of M. Boyer and N. F. Stewart [3, 4] establishes that the usual Hausdorff distances between sets are not sufficient to yield an acceptable definition of *'same form'*, but refines this metric to do so. N. F. Stewart [22] used tame homeomorphisms [12] to establish a sufficient formal condition for the informally understood requirement of *'same form'*. While Stewart's result is a significant advance, its principal shortcoming is that the original object S must be in the class of ball-like polyhedral objects, that is, the object must be a polyhedron, whose boundary is homeomorphic to a 2-sphere. The prominence of tame homeomorphisms within geometric topology is seen by noting that the well-known Alexander horned sphere was developed as a counterexample to the erroneous claim that all homeomorphs of a 2-sphere within R^3 must be tame homeomorphs. The basis for Stewart's restriction to ball-like polyhedral objects lies in his dependence upon the Three Dimensional (3D) Piecewise Linear (PL) Schoenflies Theorem [2], which is restricted to such objects. Although the 3D PL Schoenflies Theorem is an effective tool, its high level of abstraction obscures details specific to the polyhedral tolerance problem within mechanical design. Even though mathematical history is rich in investigation of variants of the 3D PL Schoenflies Theorem [2], none of those directions appear to be appropriate for the broad

[12] A tame homeomorphism between subsets of R^n is one that extends to a homeomorphism of R^n to itself.

scope of the CAGD objects under consideration. The critical aspect of the proof of the 3D PL Schoenflies Theorem lies in the development of a neighborhood about a PL 2-sphere, such that a homeomorphism of the sphere can be constrained to be the identity in the complement of this neighborhood.

In particular, N. F. Stewart [22] demonstrates that if the *'same form'* of a ball-like polyhedron is to be preserved, then it is sufficient to restrict all perturbations of its vertices to a distance less than half the minimum of all pairwise distances between its disjoint vertices, edges and faces, while simultaneously retaining co-planarity of vertices that were originally co-planar. N. F. Stewart labels this critical separation parameter as μ and his results are directly dependent upon the specified tolerance being appropriately related to μ. The computation of μ and its relation to a specified tolerance by N. F. Stewart reflects a specific engineering perspective that would be absent for most mathematicians in search of generalized theorems. N. F. Stewart narrowly restricted his domain, so as to directly use the neighborhood of the sphere developed in the 3D PL Schoenflies Theorem. While the 3D PL Schoenflies Theorem has been a long standing focus of great mathematical interest, it is possible that this new engineering perspective on its applications may stir renewed mathematical interest in its theoretical foundations.

4. Algebraic Topology and Validity of Solids

The final example is a novel application of algebraic topology to CAGD [11]. Here, the mathematical exposition is particularly terse, but the necessary background is readily available in the standard texts [6, 21]. Specifically, the Betti numbers are well known to be topological invariants. In order to adapt the Betti numbers to a useful application within CAGD, the **mod p Betti numbers**, $b_0(p), b_1(p)$, and $b_2(p)$ (where p is understood to be prime), are utilized. D. A. Lear's fundamental proposition [11] follows.

Proposition: For each prime p and for any finite CW-complex embedded in R^3, its first three Betti numbers are equal to its corresponding mod p Betti numbers; that is, for each prime p, $b_0 = b_0(p)$, $b_1 = b_1(p)$, and $b_2 = b_2(p)$, where b_0, b_1 and b_2 represent the usual Betti numbers.

The mod p Betti numbers rely upon the computation of the rank of an integer valued matrix using mod p arithmetic. The important special case to consider is when $p = 2$. The necessary matrix rank is then efficiently computed by replacing all odd entries by 1's and all even entries with 0's, and column reducing the resultant matrix. This binary arithmetic allows for efficient computer implementation via bitwise operations.

Within a CAGD environment, there is concern that an unintended error may result in the generation of an invalid solid. Such an error could arise as operator error or be a manifestation of software *'bugs'*. Because of the complexity of the CAGD software, it is valuable to have software tools available to readily address some nagging validity questions. The ability to rapidly compute these Betti numbers permits timely answers to some topological concerns, but, in general, even such information is only partial data in addressing whether two objects are topologically equivalent. Their current usefulness is to help discriminate that two objects cannot be topologically equivalent because their calculated topological invariants are not equal.

Among the concerns that could be addressed are

1. Are there holes in the object?
2. Is the object a manifold?
3. Is this object topologically identical to another?
4. Are there any self-intersections?

The specific representations of CAGD allow restriction of attention to the manageable class of finite two dimensional CW-complexes embedded in R^3, upon which the mod 2 Betti numbers can be readily computed. Extensions of this work could lead to more refined validity checks by the discovery of computationally efficient methods to determine the first homotopy group.

5. Conclusions

The intent within this article has been to illustrate the range of topological applications within CAGD. Results from the engineering community have been expressed in language that is (hopefully) accessible to mathematicians. In that endeavor, the examples given of our own work have been extended beyond their previously published limits. While no similar extending claims can be made regarding this presentation of the work of others, it is hoped that this article will serve as a catalyst for the topological community to investigate the rich application opportunities within CAGD. That investigative process can be expected to lead to further enrichment of underlying mathematical theories. Some possible avenues for such explorations have been indicated, particularly with respect to the topological/algebraic interplay and to the 3D PL Schoenflies Theorem.

In preparing this article, the authors have discovered that there is a common abstraction between the cited works of D. W. Rosen and T. J. Peters and those of M. Boyer and N. F. Stewart. While M. Boyer and N. F. Stewart were primarily interested in tolerance and D. W. Rosen and T. J. Peters were primarily interested in the design/manufacturing interface, their mathematical techniques bear some similarity. The tolerance work specified same form via tame homeomorphisms and Lipschitz constants, whereas the interface work designated equivalent form by equality of feature types. Furthermore, within this approach to tolerancing, differences were based upon a Hausdorff metric, whereas the interface results utilized a bounded taxi-cab metric. The form and metric issues suggest a commonality that may merit further investigation in attempts to understand tolerances across this complex design/manufacturing interface-especially as tolerances are paramount with respect to the design-manufacture transformation.

Any article of this brevity cannot be intended to be a comprehensive survey. Our judgement for examples to include and exclude necessarily reflects our own perspectives upon the field. There are other applications that could have served equally well to convey this range, but we believe that our examples covering set-theoretic topology, point-set topology, geometric topology and algebraic topology are representative of the range. A longer article, which will attempt to be a more comprehensive survey, is under preparation by the authors. However, it is likely that article will appear within the engineering literature. We would hope that the interest stimulated by this article would encourage mathematicians to pursue such engineering literature.

6. Acknowledgements

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