Ambient Isotopic Approximations for Surface Reconstruction and Interval Solids

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ABSTRACT

Given a nonsingular compact 2-manifold F without boundary, we present methods for establishing a family of surfaces which can approximate F so that each approximant is ambient isotopic to F. The current state of the art in surface reconstruction is that both the theory and practice are limited to generating a particular piecewise linear (PL) approximation. The methods presented here offer broader theoretical guidance for a rich class of ambient isotopic approximations. They are also used to establish sufficient conditions for an interval solid to be ambient isotopic to the solid it is approximating.

The methods are based on *global* theoretical considerations and are compared to existing *local* methods. Practical implications of these methods are also presented. For the global case, a differential surface analysis is performed to find a positive number ρ so that the offsets $F_o(\pm\rho)$ of F at distances $\pm\rho$ are nonsingular. In doing so, a normal tubular neighborhood, $F(\rho)$, of F is constructed. Then, each approximant of F lies inside $F(\rho)$. Comparisons between these global and local constraints, as well as with other methods are discussed.

Keywords

Ambient isotopy; surface reconstruction; interval solid; offsets and deformations; reverse engineering

1. INTRODUCTION AND MOTIVATION

The problem of approximation of surfaces is of fundamental importance both in theoretical as well as in applied mathematics. In particular, in computer-aided geometric design (CAGD) it plays a crucial role in the discretization of the data for meshing applications. These applications generate, in general, piecewise linear (PL) approximations of the actual surface. In order for these approximations to be of practical, as well as of theoretical, value, it is often desirable to be within a user specified tolerance. Simultaneously, it is often desirable to have topological equivalence via an ambient isotopy between the approximant and the original surface.

In this paper we propose global methods for creating a family of approximating surfaces to a given non-singular compact surface F, while ensuring that each approximant is ambient isotopic to the original surface F. While most existing reconstruction methods only provide for piecewise linear approximations, the proposed method is suitable for higher order approximations. Our primary tool is creating the offset of a surface. This is motivated by the use of offsets is the recent work of Wallner, Sakkalis et al, [27] in which some of the geometric and algebraic properties of offsets are explored. An *offset surface* is used in the construction of a normal tubular neighborhood of a manifold. Offset surfaces also have diverse potential applications in geometric modeling, such as in the definition of tolerance zones, the generation of tool paths for numerical control machining, etc. A second application is to interval solids, where sufficient conditions are given for an interval solid to be ambient isotopic to the solid that it is being approximated.

The paper is organized as follows: In Section 2, we summarize related work. In Section 3 we present some fundamental definitions and basic results from differential topology and offset surfaces. The method presented is based upon constraining the approximant to lie within a bounded offset of the surface, as determined by considerations from differential topology. To do so, a focal point is defined in Section 3.1, and an intuitive explanation of this definition is given. Section 3.2 provides the constraints that are used to define a nonsingular offset surface and a tubular neighborhood, as tools in the approximation process. In particular, we give conditions on ρ so that the offset(s) $F_o(\pm\rho)$ of F are nonsingular. Section 4 gives a formal definition of ambient isotopy and contains the main theorem, giving criteria for a family of approximating surfaces to be ambient isotopic to a single surface. This section concludes with a discussion of how these results can be applied to improve the state of the art for surface reconstruction. Section 5 expands the topic to solids, by consideration of a surface bounding a finite volume. The particular focus is upon interval solids. In Section 6 a comparison between our methods and previously published techniques is presented. Closing remarks are given in Section 7.

2. RELATED WORK

Classical aspects of topology [28] are emerging as valuable tools in solid modeling. As approximation is unavoidable in solid modeling, the question of whether an approximation is "good enough" to preserve the essential features of the object is of central importance. Previous publications by these two authors have invoked the concept of ambient isotopy, a topological notion of equivalence for admissibility of approximations to curves [20] and surfaces [5].

The notion of 'computational topology' [15] has been proposed primarily as the merging of combinatorial topology and computational geometry. Most work in computational topology to date [15; 13] has ignored differentiability and approximation. To the contrary, the present work emphasizes the integration of general topology, differential topology and approximation.

The following description is the basis for prefering ambient isotopy for topological equivalence versus the more traditional equivalence by homeomorphism [28], and summarizes the justification previously presented [5]. While a formal definition of ambient isotopy is provided in Section 4, for present purposes the following informal notion will be sufficient. Intuitively, two closed curves will not be ambient isotopic if they form different knots, which can only be converted into each other by untying one knot and retying it to conform to the other. Although any two simple closed planar curves are ambient isotopic, Figure 1 shows two simple homeomorphic space curves, which are not ambient isotopic, because they describe different knots[1]. The smooth curve depicts the simplest closed curve, known as the unknot. The PL curve is an approximation of the smooth curve. In the right half of Figure 1 the z coordinates of some vertices are specifically indicated to emphasize the knot crossings in $\mathbf{R}^3$ (All other end points have $z = 0$). All end points of the line segments in the approximation are also points on the original curve. Having this knotted curve as an approximant to the original unknot would be undesirable in many circumstances, such as graphics and engineering simulations. Similar pathologies can happen in approximating surfaces, but the work presented here can prevent these difficulties by appropriately constraining the approximations produced.

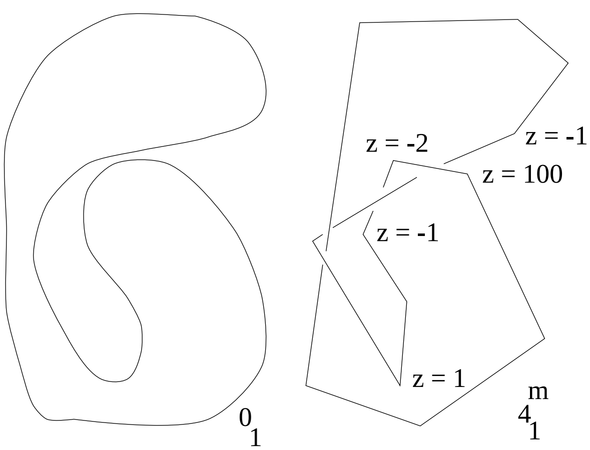

Figure 1: Nonequivalent Knots

Earlier surface reconstruction algorithms guaranteed topological equivalence to the original surface by means of homeomorphisms [3; 2; 17].

[1]The different knot classifications of 0_1 and 4_1^m are indicated.

A common technical tool for demonstrating an ambient isotopy of compact support is a type of function known as a 'push' [9]. A generalization of a push is used in the proof given in this paper.

The issue of rigorous proofs for the preservation of topological form in geometric modeling appears to have been first raised regarding tolerances in engineering design [10; 11; 26], but these papers did not specifically propose ambient isotopy as a criterion. The class of geometric objects considered was appreciably expanded by theorems for ambient isotopic perturbations of models with spline boundaries [7; 8]. For the simpler case of polygonal models, similar topology preserving approximations had been presented earlier under different technical terminology [6; 12].

In response to the example of Figure 1, a theorem was published for ambient isotopic PL approximations of 1-manifolds [20]. The proof utilizes 'pipe surfaces' from classical differential geometry [23]. The improved approximation is shown in Figure 2. There is also a of comparison of curves to α-shapes [14] via ambient isotopies [24].

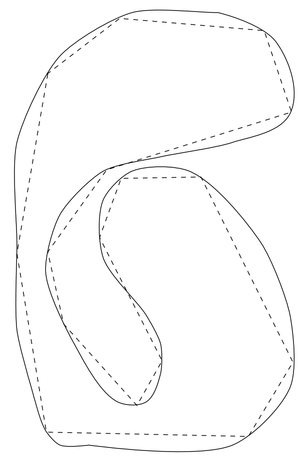

Figure 2: Ambient Isotopic Approximation

3. PRELIMINARIES

In this section we review some elementary facts about manifolds and offsets, taken from [21; 27], that we will need in the next section. In particular, we prove our first basic result, Theorem 3.2, which is the key to the proof of Corollary 3.4. The theoretical results presented here were motivated by the pragmatic view that other surface reconstruction methods [1; 4; 5] depend upon a supporting computation of the medial axis, which must also be approximated from sampled data. The methods given here eliminate this intermediate approximation of the medial axis, which is often a formidable task. This may result in a need for finer sampling, but it is beyond the scope of the present paper to make a comprehensive comparison of the detailed implementations of these competing methods. The sampling criteria presented here remains admittedly implicit. Namely, if one wishes to reconstruct a surface from sampled data, then the surface must be sampled sufficiently finely so that the reconstructed approximant lies within $F_o(\pm\rho)$. This is consonant with methods previously developed for 1-manifolds [20].

3.1 Differential Topology: Critical Points

Let $f(x_1, \cdots, x_n) : \mathbf{R}^n \to \mathbf{R}$ be a smooth function. A point $a \in \mathbf{R}^n$ is called a *critical* point of f if all first partial derivatives of f are zero at a; that is $\frac{\partial f}{\partial x_i}(a) = 0$, for all i. Furthermore, a is called a *nondegenerate* critical point of f, if

1. a is a critical point of f, and
2. The matrix $Hf(a) = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)$ of the second partial derivatives of f at a is invertible. (This matrix is also known as the *Hessian* matrix of f.)

REMARK 3.1. *A critical point of f that is not nondegenerate is called degenerate.*

The notions of degenerate/nondegenerate critical points can be carried out on a real function on a manifold W of any dimension. Here, however, we shall specialize on manifolds of dimension 2. Let then $F \subset \mathbf{R}^3$ be a manifold of dimension 2, and let $g : F \to \mathbf{R}$ be a smooth function. A point $a \in F$ shall be called a critical point of g if the gradient vector $(\frac{\partial g}{\partial x_1}, \frac{\partial g}{\partial x_2}, \frac{\partial g}{\partial x_3})$ is parallel to the unit normal vector $\mathbf{n}_a$ of F at a. Let a be a critical point of g, and let u_1, u_2 be local coordinates on F around a. Let $Hg(u) = \left(\frac{\partial^2 g}{\partial u_i \partial u_j}\right)$ be the Hessian matrix of g at a with respect to the coordinates u_1, u_2. Then, a is called degenerate/nondegenerate if $Hg(u)$ is singular/nonsingular, respectively.

Let $F \subset \mathbf{R}^3$ be as above and let $N \subset F \times \mathbf{R}^3$ be defined as

$$N = \{(q, v) \mid q \in F, \quad v \text{ is perpendicular to } F \text{ at } q\}$$

It is not difficult to see that N is a 3-dimensional manifold differentiably embedded in $\mathbf{R}^6$.

Let $E : N \to \mathbf{R}^3$ be defined as $E(q, v) = q + v$. (E is called the "endpoint" map.)

DEFINITION 3.1. *A point $e \in \mathbf{R}^3$ is a focal point of (F, q) with multiplicity μ if $e = q + v$ where $(q, v) \in N$ and the Jacobian of E at (q, v) has nullity $\mu > 0$. The point e will be called a focal point of F if e is a focal point of (F, q) for some $q \in F$.*

Intuitively, a focal point of F is a point of $\mathbf{R}^3$ where nearby normals intersect. Now consider the normal line L to F that goes through q and consists of all points $q + t\mathbf{n}_q$, where $\mathbf{n}_q$ is the unit normal of F at q and $t \in \mathbf{R}$. We then have:

LEMMA 3.1. *([21], p. 34, Lemma 6.3) The set of focal points of (F, q) along L is a subset of the points $q \pm K_i^{-1}\mathbf{n}_q$, where $i = 1, 2$, $K_i \neq 0$ are the principal curvatures at q.*

3.2 Offset surfaces

Let $F \subset \mathbf{R}^3$ be an orientable[2] 2-manifold which is C^2 (at each point of the manifold, the second derivative exists and is continuous); we will also call F a nonsingular surface in $\mathbf{R}^3$. We shall fix an orientation on F as follows: Let $v \in F$ and let v_1, v_2 be a positively oriented basis for the tangent space TF_v, regarded as a subspace of $\mathbf{R}^3$. We say that F has the *positive* orientation at v if $\det[\mathbf{n}_v, v_1, v_2] > 0$, where $\mathbf{n}_v$ is the unit surface *normal* of F at v. F has the positive orientation, if it has the positive orientation at each of its points. (The term of *negative* orientation is defined similarly, with the obvious change of sign.) Let $\rho \in \mathbf{R}$. Then, the positive offset $F_o(\rho)$ of F and the negative offset $F_o(-\rho)$ of F are defined, respectively as:

$$F_o(\rho) = \{x + \rho\,\mathbf{n}_x \mid x \in F\} \tag{1}$$

and

$$F_o(-\rho) = \{x - \rho\,\mathbf{n}_x \mid x \in F\}. \tag{2}$$

A simple geometric interpretation of the offset $F_o(\rho)$ is the surface locus swept out by the center $x + \mathbf{n}_x$ of a sphere of radius ρ as the sphere rolls over every point x of F.

In this subsection we will prove several results concerning offsets. In particular, we will give *necessary* and *sufficient* conditions on ρ so that $F_o(\rho)$ is *nonsingular* in terms of certain differential and geometric considerations of the surface F. We will restrict our attention to compact manifolds *without* boundary.

Our first result comes as a direct application of Lemma 3.1.

PROPOSITION 3.1. *Let $F_o(\rho)$ be as in (1) and let*

$$g : F \to F_o(\rho) \subset \mathbf{R}^3, \quad g(x) = x + \rho\,\mathbf{n}_x \tag{3}$$

Then, the Jacobian of g at x has nullity $\mu > 0$ if and only if $x + \rho\,\mathbf{n}_x$ is a focal point of F.

Proof: The proof is a slight modification of the proof of Lemma 3.1. ■

Proposition 3.1 shows that $F_o(\rho)$ is locally a 2-dimensional manifold at $x + \rho\,\mathbf{n}_x$ precisely when $x + \rho\,\mathbf{n}_x$ is not a focal point of F.

We will proceed with the question when g is *globally* 1-1. For this, we first define the map

$$G : F \times F \to \mathbf{R}, \quad G(x, y) = \|x - y\|^2 \tag{4}$$

Obviously, $G(x, y) > 0$, for $x \neq y$. Second, note that G has a critical value $r > 0$ since $F \times F$ is compact. Let now $x, y \in F$ with $G(x, y) = r$. Then, it is easy to see that $\mathbf{n}_x = \pm\mathbf{n}_y$ and the vector $x - y$ is parallel to both $\mathbf{n}_x$ and $\mathbf{n}_y$. Third, we claim that if

$$c = \inf\{r > 0 \mid r, \ r \text{ is a critical value of } G\} \tag{5}$$

then c is positive. For if c were to be zero, we would then have that, for each arbitrarily small positive δ there exists a pair of points (x_δ, y_δ) so that

- $\|x_\delta - y_\delta\| < \delta$,
- The vector $x_\delta - y_\delta$ is parallel to both $\mathbf{n}_{x_\delta}, \mathbf{n}_{y_\delta}$

Since δ can be arbitrarily small, then x_δ, y_δ have to belong to the same component of F. Now let a be any point of F. Using the implicit function theorem, we may choose local coordinates u_1, u_2 around a so that F becomes the graph of a smooth function $v = h(u_1, u_2)$ with $(0, 0, h(0, 0)) = a$

[2] For the context of this paper, we are considering 2-manifolds embedded in $\mathbf{R}^3$. Since non-orientable 2-manifolds without boundary can only be embedded in $\mathbf{R}^n$, for $n \geq 4$ [19, Theorem 4.7], the additional assumption of orientability leads to no loss of generalization in the present proofs.

and $\frac{\partial h}{\partial u_1} = \frac{\partial h}{\partial u_2} = 0$ at $(0,0)$. For a point $b \in F$ near a we find a corresponding point (b_1, b_2) close to $(0,0)$ so that $b = (b_1, b_2, h(b_1, b_2))$. The normal vector of M at a is $(0,0,1)$ while the normal of F at b is $(\frac{\partial h}{\partial u_1}, \frac{\partial h}{\partial u_2}, 1)$, where the partials are evaluated at (b_1, b_2). Then,

$$b - a = (b_1, b_2, h(b_1, b_2) - h(0,0)),$$

and thus the vector $b - a$ is *not* parallel to $(0,0,1)$, which is the normal of F at a, for $a \neq b$. This proves the claim.

DEFINITION 3.2. *Let $x, y \in \mathbf{R}^3$, and $X, Y \subset \mathbf{R}^3$. We define*

- $d_M(X,Y) = \max_{x \in X} \min_{y \in Y} \|x - y\|$,
- *For $a \in \mathbf{R}^3$, a point $s \in X$ is a nearest point on X to a (Please see Figure 3.) if*

$$\|a - s\| = \min\{\|a - t\| \mid t \in X\}.$$

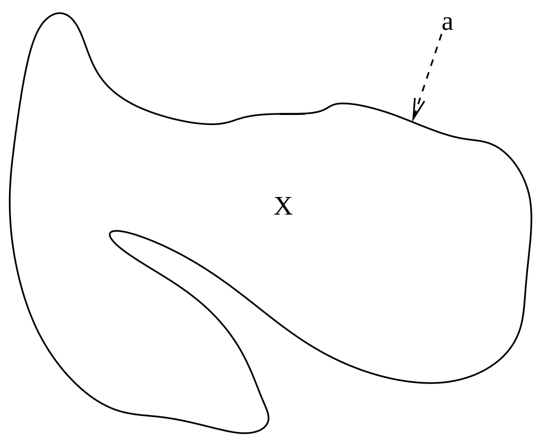

Figure 3: Nearest Point on X to a

We now consider a $\rho > 0$ so that

C1. For each point x of F neither principal curvature of F at x is equal to $\pm 1/\rho$, and

C2. $2\rho < c$, where c is as in (5).

We then have the following:

THEOREM 3.2. *For ρ as above, and g as in (3), g is an 1-1 map.*

Proof: For a positive number ϵ we may define the open set

$$F(\epsilon) = \{x \in \mathbf{R}^3 \mid d(\{x\}, F) < \epsilon\}$$

Using the ϵ-Neighborhood Theorem, [16], page 69, we may find an ϵ with the following properties:

- Each point $w \in F(\epsilon)$ possesses a *unique* nearest point in F, denoted by $\pi(w)$, and
- The map $\pi : F(\epsilon) \to F$, $w \to \pi(w)$, is a submersion

Now note that if ρ is chosen to satisfy **C1, C2** and is less than ϵ, then g is 1-1. For if $z \in F_o(\rho)$ then its distance from F is exactly ρ. Thus, if $z = x + \rho\,\mathbf{n}_x = y + \rho\,\mathbf{n}_y$, then $\|z - x\| = \|z - y\| = \rho$. Therefore, $\pi(z) = x = y$.
Suppose now that ρ is the *first* positive number for which g is not 1-1. Let then $x \neq y$ for which $g(x) = g(y)$; that is $x + \rho\,\mathbf{n}_x = y + \rho\,\mathbf{n}_y$. Then, $x - y = \rho\,(\mathbf{n}_y - \mathbf{n}_x)$. We claim that in this case (x, y) is a critical point of G. For if not, the locally 2-dimensional manifold $F_o(\rho)$ has a non-tangential self-intersection at $g(x)$, and that contradicts the choice of ρ. Thus, $\|x - y\| = 2\rho$. But $\|x - y\| \geq c$ and this is a contradiction to **C2**. ■
Finally, we have:

COROLLARY 3.3. *Let F be a compact orientable surface without boundary. Then $F_o(\rho)$ is a nonsingular surface if ρ satisfies conditions* **C1** *and* **C2**.

COROLLARY 3.4. *Let $\rho \in (0, \infty)$ be so that ρ satisfies conditions* **C1** *and* **C2**. *Then, the open set $F(\rho)$ is a* normal tubular neighborhood *of F. In addition, if r is such that $0 \leq r \leq \rho$, then,*

- *The offsets $F_o(r)$ and $F_o(-r)$ are nonsingular.*
- *Every point $w \in F(\rho)$ has a unique nearest point $\pi(w)$ in F.*
- *Let π_r be the nearest point function $\pi_r : F(|r|) \to F$, $\pi_r(v) = \pi(v)$. Then, for every point $x \in F$ the set $\pi_r^{-1}(x)$ is equal to $(x - \mathbf{n_x}, \mathbf{x} + \mathbf{n_x})$.*

The concept of a tubular neighborhood of a submanifold without boundary is not new; a result similar to the above, stating its existence, appears in [19, Theorem 5.2]. Here, however, an explicit numerical bound on the size of the neighborhood is presented, which is useful in computational applications. One can visualize a tubular neighborhood of F as follows: Suppose that F is made out of thin rubber. Then, by uniformly inflating and deflating the *interior* of F so that no singularities occur in F, a tubular neighborhood is nothing but the union of the volumes created by the inflation and deflation of F.

4. AMBIENT ISOTOPIC APPROXIMATIONS

Many approximation schemes for surfaces are concerned with the existence of a homeomorphism between the actual surface and its approximant. However, the latter does not guarantee that the surface and its approximant have the same embedding within $\mathbf{R}^3$. As an example of different embeddings in $\mathbf{R}^3$, consider the standard torus and a variant of it. Let T denote the regular torus $T = S^1 \times S^1$, where S^1 is the unit circle. Let KT denote a knotted torus, $KT = S^1 \times K$, where K is a trefoil knot, chosen so that KT is homeomorphic to T. However, within $\mathbf{R}^3$, one cannot continuously deform T into KT. This is demonstrated by showing that the spaces $\mathbf{R}^3 - T$ and $\mathbf{R}^3 - KT$ do not have the same homotopy type [22, p. 103, Theorem 1].
Two related notations are defined here for use within the rest of the paper. Both concepts are standard in general topology and can be found in any basic topology text [28].
For any topological space X and any subset A in X, the notation $cl_X A$ refers to the closure of A in X. For any

topological space X and any subset A in X, the notation $Int_X A$ refers to the interior of A in X. In cases where the particular space X being used is obvious from context, then the subscript is typically deleted, as is done in this paper, where $X = \mathbf{R}^3$.
The following definition (Please see Figure 4.) is central to the rest of the paper and it gives precise meaning to when two objects are both homeomorphic and have the same embedding within $\mathbf{R}^3$.

DEFINITION 4.1. *Let X, Y be subsets of $\mathbf{R}^3$. Then we say that X and Y are* ambient isotopic *if there is a continuous function*

$$H : \mathbf{R}^3 \times [0,1] \to \mathbf{R}^3$$

such that for each $t \in [0,1]$, $H(\cdot\, ,t)$ is a homeomorphism from $\mathbf{R}^3$ onto $\mathbf{R}^3$, $H(\cdot\, ,0)$ is the identity and $H(X,1) = Y$.

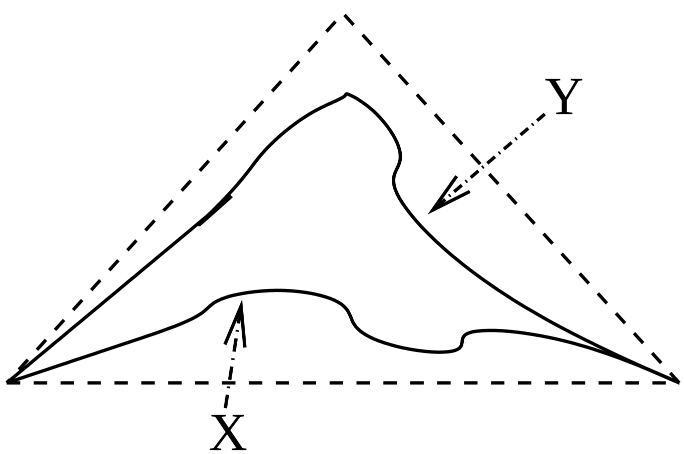

Figure 4: Ambient Isotopy

In the illustrative Figure 4, the set X is continuously deformed into Y, while simultaneously, the region bounded by X and the horizontal dashed line segment is being deformed into the region bounded by Y and the horizontal dashed line segment. Thus, the smaller region is being stretched into a larger one. At the same time the region bounded by X and the two non-horizontal line segments is begin deformed into the region bounded by Y and the two non-horizontal line segments, contracting a larger region into a smaller one. These complementary expansions and contractions allow for the definition of a continuous map that fixes all points on the boundary of and exterior to the triangle.
The techniques of offsets and tubular neighborhoods given in the previous section enable us to establish sufficient criteria for ambient isotopy of certain surfaces. In that regard, we term our results as *global* in the sense that the surface is perturbed globally.
Let F be our surface and let ρ be as in Corollary 3.4. For a point $x \in F$, let $I(x,\rho)$ be the closed line segment that connects the points $x - \rho\mathbf{n}_x$ and $x + \rho\mathbf{n}_x$. The following is the main result of this section, and provides a means of establishing a family of surfaces ambient isotopic to F.

THEOREM 4.1. *Let $F \subset \mathbf{R}^3$ be a compact nonsingular 2-manifold without boundary, and ρ as in Corollary 3.4. Let also $W \subset \mathbf{R}^3$ be a compact 2-manifold without boundary that satisfies the following:*

- *$W \subset F(\rho)$, and*
- *For every $x \in F$, $I(x,\rho)$ intersects W at precisely one point.*

Then, W is ambient isotopic to F and $d_M(F,W) < \rho$. (Please see Figure 5.)

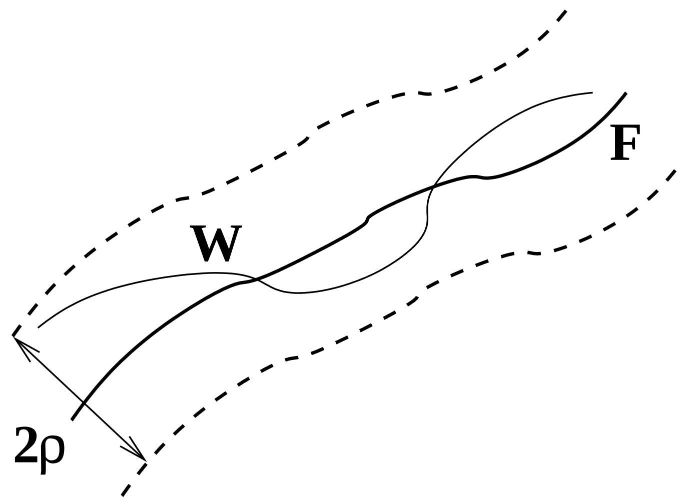

Figure 5: Intersecting W in Precisely One Point

Proof: Since F is compact, it is clear that $\partial F(\rho)$ is compact. Furthermore, since $F(\rho)$ is open and $W \subset F(\rho)$, it is also clear that $d(W, \partial F(\rho)) > 0$. Hence, for any $r > 0$, such that $r < \rho$ and $W \subset F(r)$, there exists an ϵ such that $r + \epsilon < \rho$.
For any $x \in cl_{\mathbf{R}^3} F(\rho)$, let $w_x = W \cap I(x,\rho)$ and define the function $h_{x,1} : cl_{\mathbf{R}^3} F(\rho) \times [0, 1/2] \to \mathbf{R}^3$ as follows:

$$(x, 0) \mapsto x + (r-\epsilon)\mathbf{n}_x \text{ and } (x, 1/2) \mapsto w_x,$$

and for all $t \in (0, 1/2)$, $h_{x,1}$ is obtained by linear interpolation between the two above point assignments.
Similarly, For any $x \in cl_{\mathbf{R}^3} F(\rho)$, define the function $h_{x,2} : cl_{\mathbf{R}^3} F(\rho) \times [1/2, 1] \to \mathbf{R}^3$ as follows:

$$(x, 1/2) \mapsto w_x, \text{ and } (x, 1) \mapsto x + (r+\epsilon)\mathbf{n}_x$$

and for all $t \in (1/2, 1)$, $h_{x,2}$ is obtained by linear interpolation between the two above point assignments.
Then define the function $H : \mathbf{R}^3 \times [0,1] \to \mathbf{R}^3$ by

$$H(x,t) = \begin{cases} x & \text{if } x \in \mathbf{R}^3 - F(\rho), \forall t \in [0,1], \\ h_{x,1}(x,t), & \text{if } x \in cl_{\mathbf{R}^3} F(\rho), t \in [0, 1/2], \\ h_{x,2}(x,t), & \text{if } x \in cl_{\mathbf{R}^3} F(\rho), t \in [1/2, 1]. \end{cases}$$

Note that the piecewise definition of H agrees on $\partial F(\rho)$. It is obvious that H is a homotopy such that $H(x,0) = x, \forall x \in \mathbf{R}^3$ and $H(W,1) = F$. To complete the proof that H is an ambient isotopy, it suffices to show that if $H(\cdot\, ,t)$ is a homeomorphism for all $t \in [0,1]$. However, since for each $t \in [0,1]$, the continuous function $H(\cdot\, ,t)$ is the identity outside the compact set $cl_{\mathbf{R}^3} F(\rho)$ (and is also the identity along the boundary of $cl_{\mathbf{R}^3} F(\rho)$), it only remains to show that $H(\cdot\, ,t)$ is 1-1 for all $x \in cl_{\mathbf{R}^3} F(\rho)$. But, this follows easily, since for all $x, y \in F(\rho)$, with $x \neq y$, we have, by hypothesis, that the relevant linear segments are disjoint, namely,

$$[x+(r-\epsilon)\mathbf{n}_x, x+(r+\epsilon)\mathbf{n}_x]\cap[y+(r-\epsilon)\mathbf{n}_y, y+(r+\epsilon)\mathbf{n}_y] = \emptyset.$$

The bound given on the distance between W and F follows directly from the containment of W within $F(\rho)$.

This completes the proof of a demonstration of a particular ambient isotopy, where the proof presented is similar to classical arguments [19, Chapter 5], with the additional information provided here of a specific bound on the size of the neighborhood containing the ambient isotopic images, in contrast to the classical arguments merely proving the existence of some neighborhood. ■

COROLLARY 4.2. *Let F, ρ be as in Theorem 4.1. Then for every r such that $0 \leq r \leq \rho$, $F_o(r)$ is ambient isotopic to F and $d_M(F, F_o(r)) \leq \rho$.*

The above results provide an abundance of surfaces W ambient isotopic to F. Note that any such W is inside the tubular neighborhood $F(\rho)$ and within tolerance ρ from F. Existing methods [5] produce one PL approximant, whereas Corollary 4.2 provides for the existence of infinitely many ambient isotopic approximants, each with bounded deviation from F. The new methods presented here may be valuable for applications in reverse engineering, where the final desired output is often a CAGD B-rep model with spline boundary surfaces rather than just a PL approximant.

5. APPLICATION TO INTERVAL SOLIDS

Let F be our surface, as given in Theorem 4.1. It is intuitively obvious that if one generates another surface by continuous local perturbations of F, which neither create new self-intersections nor remove any existing self-intersection, the perturbed surface will be ambient isotopic to F. In this example, we will make use of this idea and approximate F by another surface which is generated by local perturbations of F. This construction yields a PL approximation and is motivated by the recent work of Sakkalis et al, [25], in which the concept of an *interval solid* is defined and some of its fundamental topological and geometric properties are proved.

Throughout this section, a *box* is a rectangular, closed parellelpiped in $\mathbf{R}^3$ with positive volume, whose edges are parallel to the co-ordinate axes. Such boxes are used to create "interval solids", as defined and discussed in [25]. Some critical supporting material from [25] will be summarized here, to keep the paper reasonably self-contained.

Let F be our surface, and assume that F is connected. Then the Jordan Surface Separation Theorem asserts that the complement of F in $\mathbf{R}^3$ has precisely two connected components, F_I, F_O; we may assume that F_I is bounded and F_O is unbounded. Let also $\mathcal{B} = \{b_j, j \in J\}$ be a finite collection of boxes that satisfies the following conditions:

C3. $\{Int(b_j), j \in J\}$ is a cover of F.

C4. Each member b of $\mathcal{B}$ intersects F generically; that is, $b \cap F$ is a non-empty closed disk that separates b into two (closed) balls, B_b^+ and B_b^-, with $B_b^+, (B_b^-)$ lying in $F_I \cup F$ $(F_O \cup F)$, respectively.

C5. For any $b_i, b_j \in \mathcal{B}$, let $b_{ij} = b_i \cap b_j$. If $Int(b_i) \cap Int(b_j) \neq \emptyset$, then b_{ij} is also a box which satisfies **C4**.

In [25], conditions **C3 - C5** were assumed. Notice that condition **C4** indicates that every $b \in \mathcal{B}$ intersects F in a natural way. See Figure 6.

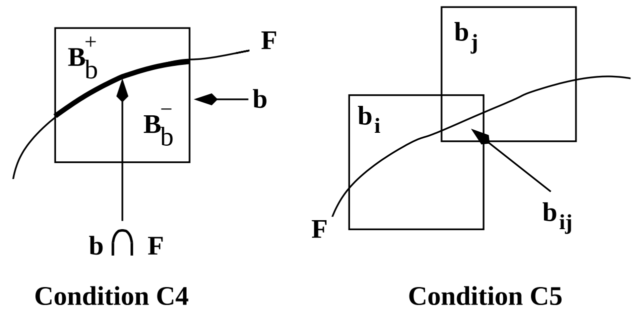

Figure 6: 2D Versions of Conditions **C4** and **C5**

The following result summarizes several previously appearing results.

THEOREM 5.1. [25, Corollary 2.1, p. 165] *If F is connected and $\mathcal{B}$ satisfies* **C3 - C5***, then $F \cup \bigcup_{j \in J} b_j$ is a solid.*

To this end, we will show that whenever F is a connected surface satisfying the hypotheses of Theorem 4.1, then F is ambient isotopic to the boundary of an interval solid constructed from F, as described in Theorem 5.1. To do so, we introduce some well known results from the literature.

DEFINITION 5.1. [9, p. 214] *A closed subset K of a PL 3-manifold-with-boundary M is* tame *if there is a homeomorphism $h : M \to M$ such that $h(K)$ is a polyhedron.*

DEFINITION 5.2. [18, p. 371] *Let M be a manifold with boundary, under the Euclidean metric d. Denote by $\mathcal{H}(M)$ the set of all homeomorphisms of M onto itself. Define a function α of $\mathcal{H} \times \mathcal{H}$ into the real extended number system as follows: $\alpha(f, g) = sup_{x \in M} d(f(x), g(x))$. Then, if $\epsilon > 0$, f and g are ϵ-isotopic if there is an isotopy H_t such that $H_0 = f$, $H_1 = g$ and if $t_1, t_2 \in [0, 1]$, then $\alpha(H_{t_1}, H_{t_2}) \leq \epsilon$.*

The following Theorem 5.2 has previously appeared as a corollary [18] to a broader result which we do not need in this paper.

THEOREM 5.2. [18, Corollary 2, p. 372] *If K is a tame compact 2-manifold in any 3-manifold M and $\epsilon > 0$, there is a $\delta > 0$ so that if h is any homeomorphism of K into M moving no point more than δ and if $h(K)$ is tame, then there is an ϵ-isotopy of M taking $h(K)$ onto K pointwise and moving no point outside an ϵ-neighborhood of K.*

We are now in a position to present our main result of this section.

THEOREM 5.3. *Let F be tame and connected. For each $\epsilon > 0$, there exists γ, with $0 < \gamma < \rho$ so that whenever a family of boxes $\mathcal{B}$ satisfies conditions* **C3 - C5***, and for each b of $\mathcal{B}$, b is a proper subset of $F(\gamma)$ (*Please see Figure 7*) then, for $S = F \cup F_I$ and $S^{\mathcal{B}} = S \cup \bigcup_{j \in J} b_j$, the sets F and $\partial S^{\mathcal{B}}$ are ϵ-isotopic with compact support. Hence, they are also ambient isotopic.*

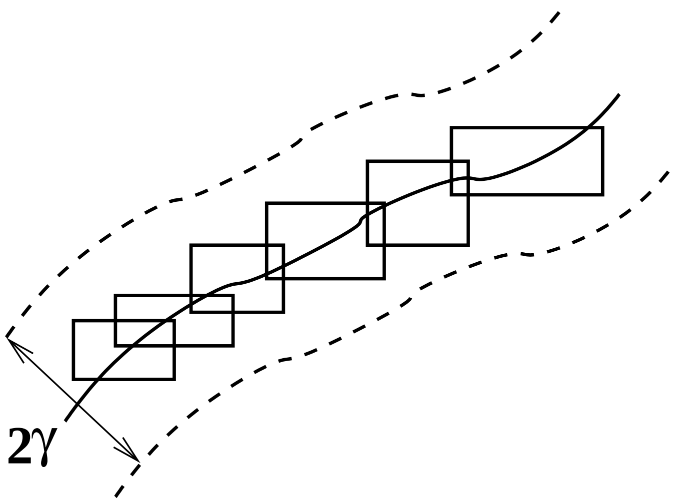

Figure 7: 2D Version of Proper Subset Condition

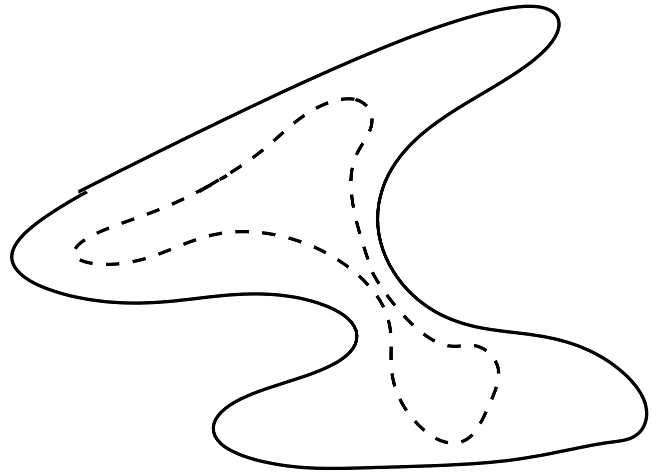

Figure 8: View of Generalized Ambient Isotopies

Proof: It has previously been shown [25] that F and $\partial S^{\mathcal{B}}$ are homeomorphic by construction of an explicit homeomorphism. It has also been shown that $M = S^{\mathcal{B}}$ is a compact 3-manifold with boundary [25]. Furthermore, it is clear that M is PL. Consider now $K = \partial S^{\mathcal{B}}$. Since K is already a polyhedron, K is tame under the identity map on M and K is also a compact 2-manifold within M. For the given ϵ, let δ be as given in Theorem 5.2. Furthermore, the homeomorphism $h : K \to F$ created [25] such that $h(K) = F$ relied upon a projection from the boundaries of the boxes onto F, where all of the boxes lay within $F(\rho)$. Hence, no point can be moved by h more than the maximal distance of the boundary of any box from F. Let now $\gamma = min\{\rho, \delta\}$. Then h satisfies the distance constraint of Theorem 5.2 and $h(K) = F$, where F is tame. Since M is compact, the ϵ-neighborhood provided by Theorem 5.2 implies that the ϵ-isotopy has compact support. Hence K and F are ambient isotopic. ■

6. GLOBAL VERSUS LOCAL METHODS

With an offset surface, there is a fixed value for the offset, resulting in a normal tubular neighborhood where the points generated as images of the endpoint map are all the same distance from the manifold. However, ambient isotopic approximations can be created where these distances need not all be equal [5], but where these distances are defined subject to local surface characteristics.

A cross-sectional view illustrating an ambient isotopy based upon local information is shown in Figure 8, where the solid curve indicates a cross-section of the original surface F and the dashed curve represents the cross-section of a surface which is ambient isotopic surface to F.

The proofs given in the present paper rely upon construction of a normal tubular neighborhood about F. While this is a sufficient condition it is *not* necessary that the original surface be entirely contained within an open neighborhood. In particular, there can be fixed points for an ambient isotopy [5].

There are two primary advantages to using offsets versus existing work [1; 5] relying upon the medial axis [5] to construct a piecewise-linear ambient isotopic approximation of a manifold:

1. approximating the medial axis is a difficult task [4] whereas the method presented here requires no such computation and
2. the ambient isotopic approximants presented here need not be PL, and this may be valuable in engineering problems where the primary data representations are free-form surfaces.

The primary dis-advantage to using the offset methods presented here is that they rely upon global, not local, bounds, so they may be overly restrictive.

Example: Let E be any non-degenerate, non-circular ellipse, which, without loss of generality, is assumed to be symmetric about the origin in the usual $x-y$ plane. There exists a minimal value of $\rho > 0$ such that the internal offset E by ρ is self-intersecting, and designate this offset by $E_o(-\rho)$. Now, consider any non-degenerate circle C centered about the origin so that C is inside $E_o(-\rho)$. There exists an ambient isotopy from E to C, which can easily be constructed by parametrizing C and E by $s \in [0, 2\pi]$ so that the mapping of points with the same parametric values is $1-1$, as depicted in Figure 9. The obvious generalizations can be made in three dimensions, leading to the conclusion that if one only wishes to generate an ambient isotopic approximation, that the bounds previously given in this paper as well as in previous work [1; 5; 20] can be overly restrictive.

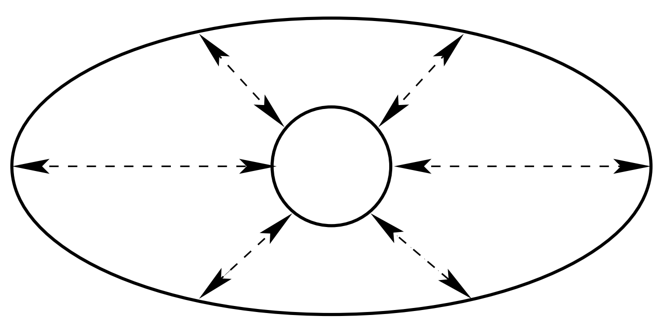

Figure 9: Ambient Isotopy by Parameter Matching

7. CONCLUDING REMARKS

In this paper we focused on a method for establishing surfaces which can approximate a given nonsingular compact manifold F without boundary so that each approximant is ambient isotopic to F. The approximants need not be piecewise linear and these non-PL approximations may be particularly useful in engineering applications where spline geometry dominates.

The results presented here start from considerations of curvature. While local values of curvature and the medial axis are closely related, examples are given to show that this primary attention to curvature affords advantages over previous methods which relied upon the medial axis, even when the objective is to create PL approximants. A specific application is the demonstration of sufficient conditions for an interval solid to be ambient isotopic to the solid it is approximating.

The results presented are restricted to smooth surfaces, even while real engineering parts often are only piecewise smooth, possibly having sharp features where surface derivatives are undefined. Consideration of such sharp discontinuities in developing ambient isotopies over multi-patch spline surfaces has appeared in the literature [8], but it is beyond the scope of the present investigation to integrate these two ideas. The theory presented here is offered as an important first step to the more challenging future extensions to real engineering parts. Indeed, the present authors enjoy current funding from the National Science Foundation to pursue these issues.

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9. APPENDIX: TOPOLOGY DEFINITIONS

DEFINITION 9.1. *A function f on spaces X and Y is a* homeomorphism *if f is bi-continuous, 1-1 and onto.*

DEFINITION 9.2. *A function f from X onto itself has compact support if there exists a compact set $A \subset X$ such that f is the identity except possibly on A.*

DEFINITION 9.3. *Two functions f and g from a space X into a space Y is called* homotopic *if there exists a continuous function $F : X \times [0,1] \to Y$ such that for each point $x \in X$,*

$$F(x,0) = f(x) \quad \text{and} \quad F(x,1) = g(x).$$

DEFINITION 9.4. *Two functions f and g from a space X into a space Y is called* isotopic *if they are homotopic via a function F such that for each $t \in [0,1]$, $F(\cdot,t)$ is a homeomorphism.*

If the original functions f and g are both onto Y, then we will interchangeably refer to the functions being isotopic, as well as the spaces X and Y being isotopic. It is this latter usage that is adopted within the main body of this paper in the definition of *ambient isotopy* given in Definition 4.1.