Unknots With Highly Knotted Control Polygons

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Abstract

For a rich class of composite cubic Bézier curves, an *a priori* bound exists on the number of subdivisions to achieve ambient isotopy between the curve and its control polygon. The authors of that theorem did not present any examples when the original control polygon is **not** ambient isotopic to the curve. An example is given here of a composite cubic Bézier curve that is the unknot (a knot with no crossings), but whose control polygon is knotted. It is also shown that there is no upper bound on the number of crossings in the control polygon for an unknotted composite Bézier curve.

There can be substantial topological differences between a curve and its control polygon, as depicted in Figure 1 and explained, below. A *knot* will be considered to be

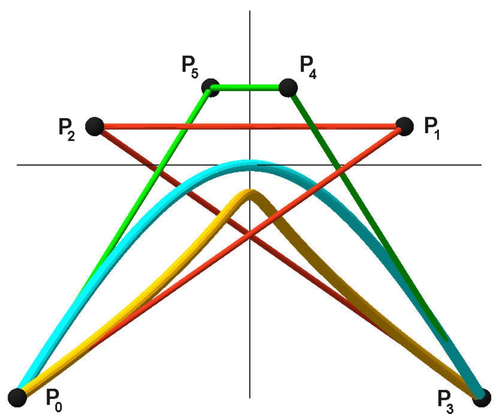

Figure 1: Unknot with Knotted Control Polygon

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a closed, non-self-intersecting curve with a specific embedding in $\mathbb{R}^3$. When such a knot is described as a composite Bézier curve[2] for analysis or detailed geometrical manipulation, it would be highly advantageous if most of the calculations and operations could be done with just the linear segments of the control polygon. One topological difference is that the control polygon can have self-intersections when the associated curve does not [2]. However, as the control polygon of any Bézier curve is subdivided, it converges to the Bézier curve.

Any non-self-intersecting C^1 composite Bézier curve with regular parameterisation will, *after sufficiently many subdivisions*, have a non-self-intersecting control polygon [3], so that the curve and its control polygon are homeomorphic[3]. The options for embedding closed curves in the plane are quite restricted so that homeomorphism is the crucial equivalence relation between closed planar curves, with the salient distinction being the presence or absence of self-intersections. However, there are significant differences in how closed, non-self-intersecting curves can be embedded within $\mathbb{R}^3$, as formally captured by ambient isotopy (Definition 0.1.) Ambient isotopy is a fundamental concept in knot theory [4]. Practical applications of ambient isotopy appear in geometric modeling, visualization and animation [5].

Definition 0.1. *A continuous function $H : \mathbb{R}^3 \times [0, 1] \to \mathbb{R}^3$ is an* **ambient isotopy** *between subsets X and Y of $\mathbb{R}^3$ if $H(\cdot, 0)$ is the identity, $H(X, 1) = Y$, and for each $t \in [0, 1], H(\cdot, t)$ is a homeomorphism from $\mathbb{R}^3$ onto $\mathbb{R}^3$. The sets X and Y are then said to be* **ambient isotopic**.

Within $\mathbb{R}^3$, there exists a rich class of non-self-intersecting, composite, cubic Bézier curves for which each curve will, *after sufficiently many subdivisions*, be ambient isotopic with its control polygon [6]. This class includes the example created here, which constructs a non-trivial polygonal knot as the control polygon of a Bézier unknot. Such examples had not been previously provided in the earlier work [6].

Example 0.1. *Denote by* $\mathbf{c}$ *the closed, composite cubic Bézier curve with control points, $P_0, \ldots P_5$, respectively listed as:*

(-6, -6, 12), (4, 1, -1), (-4, 1, 1), (6, -6, -12), (1, 2, 4), (-1, 2, -4).

Proposition 0.1. *Curve* $\mathbf{c}$ *is the unknot but the control polygon of* $\mathbf{c}$ *is a trefoil*[4].

Proof: Let K denote the control polygon of $\mathbf{c}$. This is a closed, non-self-intersecting curve embedded in $\mathbb{R}^3$. Note that K has two superfluous undercrossings:

$$[P_2, P_3] \text{ under } [P_5, P_0] \text{ and } [P_2, P_3] \text{ under } [P_0, P_1].$$

[2]For composite Bézier curves, the word 'knot' is also used for a 'junction point' [1].

[3]A continuous function $f : X \to Y$ is a **homeomorphism** between X and Y if f is bi-continuous, 1-1 and onto. The sets X and Y are then said to be **homeomorphic**.

[4]A trefoil is the knot with three crossings.

Informally, $[P_2, P_3]$ can be stretched and pulled beyond P_0 to eliminate these two undercrossings, without changing the knot type. Formally, this is the ambient isotopy of linear extrapolation of $[P_2, P_3]$ until the x co-ordinate of P_2 is sufficiently negative and the y co-ordiante of P_3 is sufficiently negative[5]. The remaining three crossings are alternating, as can be verified by linear interpolation, so that K is ambient isotopic to a piecewise linear trefoil [7].

The vertices of K are the control points for a composite cubic Bézier curve, $\mathbf{c}$. To show that $\mathbf{c}$ is the unknot, it suffices to show that there are no self-intersections in the projection of $\mathbf{c}$ into the plane $z = 0$. Denote by $\tilde{\mathbf{c}}_\mathbf{1}$ the component created from projection of the control points P_0, P_1, P_2, P_3 and by $\tilde{\mathbf{c}}_\mathbf{2}$ the component created from projection of the control points P_3, P_4, P_5, P_0. Both $\tilde{\mathbf{c}}_\mathbf{1}$ and $\tilde{\mathbf{c}}_\mathbf{2}$ are assumed to be parameterized over $[0, 1]$. Let d_2 denote the Euclidean distance function in $\mathbb{R}^2$. Published analyses [2, Example 1.7(e)] show that the curve $\tilde{\mathbf{c}}_\mathbf{1}$ is non-self-intersecting because $d_2(\tilde{P}_1, \tilde{P}_2) < d_2(\tilde{P}_0, \tilde{P}_3)$, where these distance arguments are the projections of P_1, P_2, P_0, P_3, respectively. The curve $\tilde{\mathbf{c}}_\mathbf{2}$ is non-self-intersecting because of its convex control polygon. It remains to show that these components intersect only at their end points.

The subdivided control polygons in the half-plane $x \leq 0$ is shown with solid black lines in Figure 2. Subdividing $\tilde{\mathbf{c}}_\mathbf{1}$ and $\tilde{\mathbf{c}}_\mathbf{2}$ at $t = 0.5$ gives control points in the half-plane defined by $x \leq 0$ of

$\tilde{\mathbf{c}}_\mathbf{1}$: $(-6, -6,), (-1, -2.5),$
$(-0.5, -0.75), (0, -0.75);$

$\tilde{\mathbf{c}}_\mathbf{2}$: $(0, 0), (-1.75, 0),$
$(-3.5, -2), (-6, -6).$

These convex hulls within the half-plane $x \leq 0$ intersect only at P_0, shown in Figure 2.

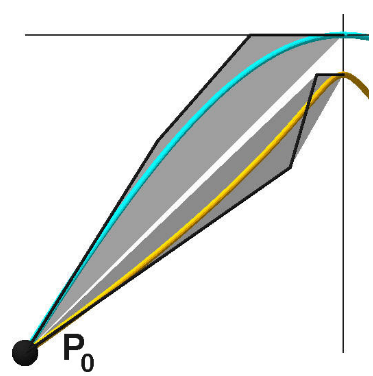

Figure 2: Subdivision & Convex Hulls

The proof is completed by using the symmetry about the y-axis. ■

Highly Knotted Control Polygons: We can now use this configuration to form an unknot with a control polygon of arbitrary knottedness.

Denote four instances of K knot of Figure 1 as K_1, K_2, K_3, K_4. For $i = 1, \ldots 4$, cut open each K_i at its initial point and then the K_i can be joined in the pinwheel pattern shown in Figure 3. Two new knots are formed; a polygonal knot with 12 crossings and a composite Bézier curve of the unknot. This process can be formalized as a connected sum of knots [4] and generalized for each $n \geq 1$ to exhibit a knotted control polygon with $3n$ crossings for a Bézier unknot. This lack of an upper bound on the

[5] This is equivalent to applying a Reidemeister move [7].

number of crossings in the control polygon leads to radically different embeddings in $\mathbb{R}^3$ compared to the corresponding Bézier unknot.

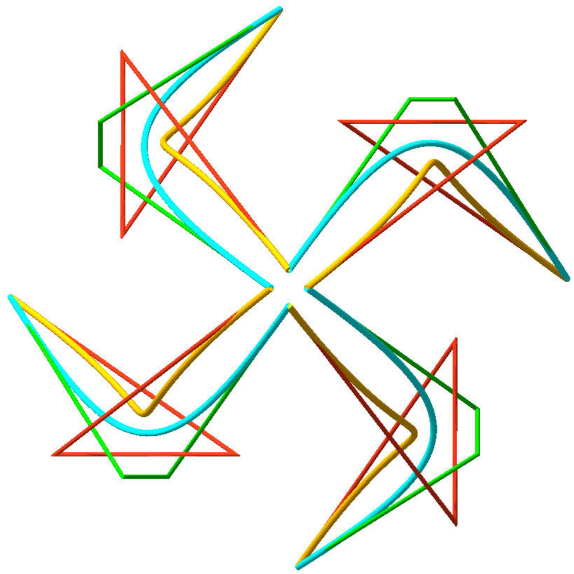

Figure 3: Connected Sum for the Control Polygon of a Bézier Unknot

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